Booklet of Abstracts

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Contents

Invited Talks

Some reflections ahead of the fortieth anniversary of the publication of J.-Y. Girard's article *Linear Logic*

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Abstract

J.-Y. Girard's article *Linear Logic*, published in 1987, marked a pivotal moment in the history of logic:

- it presented important results, and an in-depth study of classical logic and intuitionistic logic by means of new concepts and new methods;
- it paved the way to novel research on the interactions between logic and other disciplines (foremost among them, theoretical computer science);
- it significantly influenced various branches of logic (above all, proof theory);
- it has enabled (and, through its ongoing developments, continues to enable) profound reflections that enhance our understanding of the relationships between logic and philosophy, the very status of logic, and the content of some very important theorems stated in mathematical logic.

Some of these topics will be presented and discussed during the talk.

On the elementary theory of the real exponential field

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I will report on recent work with Francesco Gallinaro, available on arXiv. Gödel's incompleteness theorems imply that the elementary theory of the ring of integers is undecidable. The same holds for the theory of the field of rational numbers, since the integers are definable therein by Julia Robinson's result. By contrast, Tarski's quantifier-elimination methods established the decidability of the complete theories of the real and complex fields. The situation becomes more delicate when exponentiation is added to the language. The complete theory of the complex exponential field is undecidable, since its exponential kernel allows integer arithmetic to be interpreted.

The decidability of the complete first-order theory $T_{\exp}$ of the real exponential field $\mathbb{R}_{\exp}$ remains a long-standing open problem, but fundamental results by Wilkie and Macintyre suggest that the answer may be positive. It is known that the integers are not interpretable in this structure, thus removing one of the obstacles to decidability. More deeply, Macintyre and Wilkie showed that decidability is equivalent to the Last Root Conjecture, a weakening of Schanuel's conjecture in transcendental number theory.

Since a complete theory is decidable if and only if it is recursively axiomatizable, the decidability problem may equivalently be approached by seeking an effective axiomatization. A general strategy is to start from a categorical second-order characterization of the intended structure and replace the relevant second-order principles with first-order axiom schemes. Of course, there is no guarantee that this strategy is successful: in the case of the natural numbers, it leads from categorical second-order arithmetic to first-order Peano arithmetic, which is recursively axiomatizable but not complete. In the case of the ordered field of real numbers, however, the analogous passage leads to the first-order theory of real closed fields, which is complete, recursively axiomatizable, and therefore decidable.

These precedents suggest that a similar transformation of an appropriate second-order characterization of the real exponential field may provide a natural candidate for a recursive axiomatization of $T_{\exp}$. The second-order axiom is Dedekind completeness, and the corresponding axiom scheme is definable completeness. One is thus led to conjecture that the models of $T_{\exp}$ are the definably complete exponential fields satisfying $\exp(x) \geq 1 + x$. This remarkably simple set of axioms is proved to be complete assuming Schanuel's conjecture, the role of the latter being to ensure the existence of an archimedean model.

Temporal Synthesis as a Foundation for Strategic Reasoning in Artificial Intelligence

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Abstract

Temporal Synthesis studies the automatic synthesis of interactive programs (strategies) from declarative specifications expressed in temporal logic. In this talk, we show how Temporal Synthesis provides a principled foundation for strategic reasoning in autonomous AI systems, grounding deliberative behavior in formally specified temporal goals. The key to this research path lies in the rich body of concepts developed in reasoning about actions and planning, combined with a precise treatment of nondeterministic environments and temporal objectives. In such settings, plans must be treated as strategies rather than being blurred with individual execution traces, and goal satisfaction evolves during execution rather than being reducible to reaching states with fixed properties. These features are naturally captured within the temporal synthesis framework. Technically, we focus on synthesis from Linear Temporal Logic on finite traces (LTLf). LTLf specifications compile into deterministic finite automata (DFAs), which can be viewed as two-player game arenas, yielding efficient and scalable synthesis procedures. We then lift these finite-trace results to infinite traces through the Manna–Pnueli hierarchy, introducing LTLf+ and its obligation fragment while largely preserving algorithmic simplicity. Finally, we move beyond individual strategies to analyze the space of all strategies satisfying a specification. By characterizing the set of compliant execution traces, without assuming individual strategies to be analyzable, we provide formal foundations for guard-railing and responsibility attribution in agentic AI systems composed of multiple agents that make decisions independently.

Giuseppe De Giacomo is a Professor of Computer Science in the Department of Computer Science at the University of Oxford and a Professor in the Department of Computer, Control, and Management Engineering at Sapienza. His research spans theoretical, methodological, and practical aspects of Artificial Intelligence and Computer Science, with major contributions in Knowledge Representation, Reasoning about Actions, Generalized Planning, Autonomous Agents, Reactive Synthesis and Verification, Service Composition, Business Process Modeling, and Data Management and Integration. He is a Fellow of AAAI, ACM, and EurAI. He received an ERC Advanced Grant for the project *White-Mech: White-box Self Programming Mechanisms*. He served as Program Chair of ECAI 2020 and KR 2014. He is a member of the Board of EurAI and chairs the steering committee of EurAI's annual summer school ESSAI.

The isomorphism relation of non-Archimedean Polish groups

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Kechris, Nies and Tent proposed a program to study the isomorphism relation of Borel classes of non-Archimedean Polish groups and to determine their exact complexity in the Borel reducibility hierarchy. In the first part of this talk I will give a survey of the known results in this program. Then I will focus on showing a nontrivial lower bound for the complexity of the isomorphism relation of all non-Archimedean Polish groups. This is done by considering procountable groups, or otherwise known as TSI non-Archimedean Polish groups. In the technical theorem we show that the isomorphism relation for all procountable groups is not Borel reducible to any orbit equivalence relation induced by a Borel action of a Polish group. This is joint work with Andre Nies and Gianluca Paolini.

Bilateral Class Theory

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In previous work, I introduced a family of naïve theories of classes formulated using bilateral logic, that is a logic in which sentences are decorated with signs for assertion and rejection. As I will show, these theories can be shown to axiomatize the minimal fixed points of hierarchies obtained via a step-by-step construction of the extension and anti-extension of the membership predicate (mirroring Kripke's constructions of the extension and anti-extension of the truth predicate). Moreover, in one of these theories, we can have a suitably coarse-grained notion of identity, which has proved to be a major challenge for naïve class theories. I will then report on ongoing work with Lorenzo Rossi on the relation between naïvete and compositionality for classes in a bilateral setting. I will present a compositional class theory, again formulated in bilateral logic, and show that the theory is equivalent to the naïve theory that has the resources to solve the identity problem. Indeed, I will suggest that the identity problem is a kind of compositionality problem in a class-theoretic context.

TBA

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Iterated ideals in bqo theory

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The notion of better quasi-order (henceforth bqo) was introduced in [2] by Nash-Williams in 1965. Originally a tool to prove that an order is a *well* quasi-order (henceforth wqo), thanks to their strong closure properties and their ubiquity, bqo's have since become a very useful and central concepts in infinitary combinatorics. Bqo's are also quite interesting from the point of view of reverse mathematics: the logical strength of several results coming from bqo theory has proved rather hard to pin down. An example of this is given by Nash-Williams' result that if Q is bqo, then so is the set of transfinite sequences with range Q (ordered by embeddability): on the one hand, a formalization of the classical proof of this statement uses the so-called Minimal Bad Array Lemma, which is known to be equivalent, over ATR_0, to the strong system Π^1_2-CA_0, while on the other, some considerations on the complexity of the statement show that Nash-Williams' result cannot imply Π^1_1-CA_0 over ATR_0 (this account is not entirely precise and we refer to [1] for a better presentation of it). In general, to find better upper bounds for the strength of theorems of bqo theory, we need thus to obtain new proofs of these results, which do not use the Minimal Bad Array Lemma. The main focus of this talk is the presentation of a tool which we believe to be very helpful in this task, namely the structure of the *iterated ideals* on Q, which we call $\dot{I}^*(Q)$.

Although the "operative" condition for a quasi-order Q to be bqo is stated in terms of barriers and arrays, one gets a better intuition if instead one focuses on the *iterated powerset* of Q: as Nash-Williams already noticed, it turns out that Q is bqo if and only if the iterated powerset of Q is wqo. We will see that iterated ideals have this property as well, and we will provide a more precise relationship between the two structures.

We will then focus on the iterated ideals in the case that the order Q is of the form $P^{<\omega}$ (i.e, finite sequences with range P), for another quasi-order P. Working in (a theory similar to) ATR_0, we will see that, if we enrich $\dot{I}^*(P^{<\omega})$ with a "concatenation" operation naturally derived from the one on $P^{<\omega}$, we obtain a very nice presentation of $\dot{I}^*(P^{<\omega})$ in terms of "indecomposable" elements: this further reduction allows us to conclude that $\dot{I}^*(P^{<\omega})$ is wqo if $\dot{I}^*(P)$ is, and so, by the results stated above, that $P^{<\omega}$ is bqo if P is. Thanks to results obtained by Marcone in [1], we can conclude that Nash-Williams' theorem is provable in ATR_0.

This is joint and ongoing work with Alakh Dhruv Chopra, Fedor Pakhomov, and Philipp Provenzano.

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What if? Conditionals through Algebraic Logic

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Conditional statements are propositions of the form "If [Antecedent] then [Consequent]". Given their central role in the understanding of reasoning, they have been extensively investigated across a range of disciplines, including mathematical and philosophical logic, as well as areas of computer science and artificial intelligence such as knowledge representation and reasoning. In particular, *counterfactual conditionals*, i.e., conditional statements where the antecedent is assumed to be false, are widely used both in theory and applications as a way of modeling causality. The idea is to interpret "X is caused by Y" as "If Y had not occurred, then X would not have occurred"; e.g., "If global emissions had been reduced, the current rate of global warming would be lower".

Arguably, the most influential logical analysis of (counterfactual) conditional statements is rooted in the work of the philosopher David Lewis, with his *variably strict conditionals* [3]; importantly, these are also connected to the famous *causal models* of Pearl [1], extensively used in real-world applications. The central idea of D. Lewis, inspired also by the work of Stalnaker [7], is to use possible-world models expanded with a notion of similarity or closeness; then to evaluate a conditional "If A then C", one considers the worlds in which A holds that are most similar to the actual world, and checks whether C is true in those worlds.

While the literature analyzing Lewis's logics for conditionals is vast, their mathematical analysis has only gained traction in recent years. In particular, their logico-algebraic analysis has been developed in [5, 6]. We will discuss these works, which introduce algebraic semantics for Lewis's logics and develop their duality theory in terms of enriched Stone spaces, connecting the algebraic semantics with Lewis's models. In this discussion it will be apparent a close connection to classical modal logics and their semantics.

The works [5, 6] bring the most well-known logical approaches to counterfactual conditionals into the realm of algebraic logic. We will then explore a different direction to fill the gap between the study of conditionals and the field of algebraic logic, by discussing a conditional connective arising as a quotient operator in Boolean algebras. Surprisingly, this intuitive approach will lead us back to the classical Lewis-Stalnaker models.

Finally, we will discuss how the algebraic perspective can be extended to deal with the *probability* of conditional statements. In order to do so, one needs to go beyond classical conditional probability. Indeed, Lewis's celebrated triviality result [4] shows that, if conditionals are represented as events in the same probability space as their antecedents and consequents, then identifying the probability of a conditional with the corresponding conditional probability forces the structure to collapse into triviality. This motivates the use of the *imaging* method [4, 2] as an alternative mechanism for updating probabilities, with the aim of developing new axiomatic frameworks for reasoning with conditionals.

This talk is based on recent joint works with, respectively, Tommaso Flaminio and Francesco Manfucci, and Giuliano Rosella.

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Award Talks

A categorical duality for metrically complete lattice-ordered Abelian groups with unit

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Abstract

This extended abstract presents joint work with Vincenzo Marra and Luca Spada on a Stone–Gelfand duality for metrically complete unital lattice-ordered Abelian groups. We identify a category of compact Hausdorff spaces enriched with local arithmetic data and prove that it is dually equivalent to the algebraic category. The resulting duality simultaneously extends classical Stone duality and Yosida duality, which arise as its two extremal cases.

1 From Stone duality to Yosida duality

Dualities between algebraic and topological structures have played a central role in logic since Stone's representation theorem for Boolean algebras. Stone duality identifies Boolean algebras with algebras of clopen subsets of Stone spaces and, categorically, yields a contravariant equivalence between Boolean algebras and compact zero-dimensional Hausdorff spaces.

A rather different family of dualities arose in functional analysis. Theorems of Gelfand, Kakutani, Stone, and Yosida represent various ordered or normed algebraic structures as algebras of continuous real- or complex-valued functions on compact Hausdorff spaces. For the present discussion, the most relevant result is Yosida duality: compact Hausdorff spaces correspond contravariantly to metrically complete vector lattices with a distinguished unit.

Stone and Yosida dualities appear to sit at opposite ends of a spectrum. Stone duality concerns highly disconnected spaces and algebraic structures of a discrete character, whereas Yosida duality involves arbitrary compact Hausdorff spaces and real-valued continuous functions. Nevertheless, these two worlds coexist naturally in many-valued logic. By Mundici's equivalence, MV-algebras—the algebraic semantics of Łukasiewicz logic—correspond to unital lattice-ordered Abelian groups. These groups interpolate between the Boolean and vector-lattice cases, suggesting that Stone and Yosida duality should be extremal instances of a single more general picture.

The aim of our work [1] is to construct such a picture for metrically complete lattice-ordered Abelian groups with unit, abbreviated below as metrically complete ℓ-groups.

2 The representation problem

A unital Abelian ℓ-group is an Abelian group G, equipped with a translation-invariant lattice order and a distinguished positive unit 1. The unit determines a natural pseudometric,

$$d(v,w) = \inf\left\{\frac{p}{q} \in \mathbb{Q}_{\geq 0} : -p1 \leq q(v-w) \leq p1\right\},$$

and we consider those groups for which this is a complete metric. The standard example is $C(X,\mathbb{R})$, with the uniform metric, for a compact Hausdorff space X.

Goodearl and Handelman proved that every metrically complete ℓ-group can be represented as a group of continuous functions on a compact Hausdorff space, subject to pointwise restrictions on their values. More precisely, one obtains a compact Hausdorff space X and, for each $x \in X$, a subgroup

$$D_x \in \{\mathbb{R}\} \cup \left\{\tfrac{1}{n}\mathbb{Z} : n \geq 1\right\}$$

such that

$$G \cong C_D(X) := \{f \in C(X, \mathbb{R}) : f(x) \in D_x \text{ for every } x \in X\}.$$

The set D_x records the local arithmetic constraint imposed at x.

This theorem strongly suggests a common generalization of Stone and Yosida duality, but it is not itself a duality. There are two obstructions. First, arbitrary spaces with pointwise restrictions are not uniquely determined by the represented ℓ-group: non-isomorphic enriched spaces may yield isomorphic groups. Second, the theorem does not identify the topological maps corresponding to unit-preserving ℓ-homomorphisms. Thus the problem is to isolate both the correct spatial objects and the correct morphisms.

3 Normal arithmetic spaces

We solve the first problem by introducing *normal a-spaces*, (short for *normal arithmetic spaces*. Such an object consists of a compact Hausdorff space X together with denominator data $x \mapsto D_x$ as above, satisfying two additional requirements.

The first is a separation axiom. Whenever $x \neq y$, there exist disjoint open neighbourhoods U of x and V of y such that every point outside $U \cup V$ is unconstrained, that is, $D_z = \mathbb{R}$ for all $z \notin U \cup V$.

The second is a semicontinuity condition: for every $n \geq 1$, the subset

$$\{x \in X : D_x \subseteq \tfrac{1}{n}\mathbb{Z}\}$$

is closed. Informally, arithmetic constraints may persist at limit points but cannot become abruptly stronger there.

These axioms are precisely what is needed to make the representation canonical. They also clarify how the new spaces interpolate between the classical cases. If $D_x = \mathbb{R}$ for every x, the additional structure disappears and one recovers an arbitrary compact Hausdorff space. If $D_x = \mathbb{Z}$ for every x, the separation axiom forces clopen subsets to separate points, so that X is a Stone space.

A guiding example is the unit interval with its usual topology and its natural denominator structure:

$$D_x = \mathbb{R} \quad \text{if } x \text{ is irrational}, \qquad D_{m/n} = \tfrac{1}{n}\mathbb{Z}$$

whenever m/n is in lowest terms. This example explains the terminology and displays, within one compact space, the coexistence of continuous and discrete arithmetic behaviour.

A key technical result is an arithmetic refinement of Urysohn's lemma: continuous functions separating suitable closed sets can be chosen so as to respect all local denominator constraints. From this we obtain an analogue of the classical embedding theorem for compact Hausdorff spaces: normal a-spaces are exactly the closed subspaces of powers of the arithmetically enriched unit interval. This result is the main topological ingredient in the duality.

4 The duality

To describe the morphisms, let (X, D) and (Y, E) be normal a-spaces. A continuous map $f : Y \to X$ is called denominator-decreasing when

$$D_{f(y)} \subseteq E_y \qquad \text{for every } y \in Y.$$

This is exactly the condition ensuring that composition with f sends denominator-respecting functions on X to denominator-respecting functions on Y:

$$C_D(X) \longrightarrow C_E(Y), \qquad g \longmapsto g \circ f.$$

Our main theorem states that this construction yields a dual equivalence:

> The category of metrically complete Abelian ℓ-groups with unit and unit-preserving ℓ-homomorphisms is dually equivalent to the category of normal a-spaces and denominator-decreasing continuous maps.

In particular, every metrically complete ℓ-group has a canonical representation as $C_D(X)$ for a normal a-space (X, D); the space is determined uniquely up to isomorphism, and every unit-preserving ℓ-homomorphism is induced uniquely by a denominator-decreasing continuous map in the opposite direction.

The theorem simultaneously recovers two classical dualities. When all denominators are $\mathbb{R}$, the algebraic objects are metrically complete vector lattices and the duality reduces to Yosida duality. When all denominators are $\mathbb{Z}$, the spatial objects are Stone spaces. On the algebraic side one obtains groups of the form $C(X, \mathbb{Z})$; applying Mundici's equivalence and restricting to the unit interval $[0, 1]$ recovers the Boolean algebra of clopen subsets of X, and hence classical Stone duality.

Thus the discrete and continuous representation theories are not separate phenomena. They are the two boundary cases of a single categorical duality whose intermediate objects combine topology with local arithmetic restrictions. Besides providing the first duality for metrically complete ℓ-groups, this framework gives a geometric interpretation of metrically complete MV-algebras and clarifies the structure hidden in the Goodearl–Handelman representation theorem.

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TBA

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Borel classification of simplicial complexes and classes of non-compact manifolds

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Abstract

Classification problems are ubiquitous in mathematics. Descriptive set theory provides a way to compare their complexity via Borel reducibility, leading to a hierarchy of classification problems. In this talk I will present a line of research initiated in my PhD thesis and further developed in subsequent work, concerning how this viewpoint applies to classification problems for some geometric objects, such as manifolds, and how a new duality principle, called blurry duality, yields complete invariants for simplicial complexes up to PL-homeomorphism and for many classes of non-compact manifolds up to homeomorphism. The resulting classification problems turn out to be Borel bireducible with countable graph isomorphism. This is joint work with Vadim Weinstein.

Contributed Talks

Theory and applications of the non-commutative logic BV

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Abstract

This talk aims to present the logic BV, a non-commutative logic extending multiplicative linear logic with a non-commutative self-dual connective $\triangleleft$, and the recent results on its proof theory and applications. After recalling the original motivation for the introduction of BV and the challenges that arose in the design of its proof system, we will present some recent results on its proof theory and applications, including a logical account of process calculi, categorical models of quantum computation, and potential applications to the type theory for imperative programming languages.

1 Introduction

The logic BV was introduced by Guglielmi in the attempt of providing an inference system for Retoré's Pomset logic, which originally has only proof nets as a way to represent proofs. The challenge in the design a proof system for Pomset logic depends on the presence of its characteristic non-commutative self-dual connective $\triangleleft$, whose semantics emerges as a connective 'in between' the (commutative) multiplicative conjunction $\otimes$ and the (commutative) multiplicative disjunction $\parr$ in the study of the model of coherent spaces for linear logic.

Other than the challenges in the design of a proof system for BV (see below), the motivation for its introduction was to provide a logic with a non-commutative connective that could be used to model the sequential composition of processes, thus providing a logical framework for the study of concurrency and interleaving. In fact, the key feature of the connective $\triangleleft$ is not the fact that it is non-commutative (logic with similar connectives had been studied before, see e.g. [13, 16]), but the fact that it interacts with the other (non-commutative) connectives in a way that allows to express the fundamental law of interleaving concurrency by interpreting the $\parr$ as a parallel composition and $\triangleleft$ as a sequential composition.

$$\big((A \parr B) \triangleleft (C \parr D)\big) \multimap \big((A \triangleleft C) \parr (B \triangleleft D)\big) \tag{1}$$

This talk aims to present the logic BV and its proof system, and to give an overview of results both on its proof theory and its applications, including a logical account of process calculi, categorical models of quantum computation, and potential applications to the type theory for imperative programming languages. After recalling the language, its proof system, and some of its properties, we will present some recent results in connection with the existing literature. This talk is based on the following papers:

- [3, 1, 2] for the choreographies-as-proofs correspondence inspired by BV.
- [5] for the definition of *strong* BV*-categories*.
- [4] for the definition of IBV, an intuitionistic version of BV that can be used as a type system for imperative programming languages.

Disclaimer. The main objective of this talk is to initiate a discussion on further research directions for BV beyond the scope of this presentation. In particular, I am interested in receiving feedback and suggestions on other logics with non-commutative connectives, and on possibly related topics in algebraic logic and model theory.

2 The proof system for BV

Formulas of BV are generated by a set of atoms $\mathbb{A}$ and a unit ($\circ$) using the connectives of **tensor** ($\otimes$), **par** ($\parr$), and **seq** ($\lhd$) as follows:

$$\text{Formulas} \qquad A, B ::= \circ \mid a \mid a^\perp \mid A \parr B \mid A \lhd B \mid A \otimes B \qquad \text{with } a \in \mathbb{A} \tag{2}$$

The (linear) implication $A \multimap B$ is defined as $A^\perp \parr B$, where $A^\perp$ is the **linear negation** of A defined by extending the negation of atoms to formulas by the following **De Morgan laws**:

$$\circ^\perp = \circ \qquad (A^\perp)^\perp = A \qquad (A \parr B)^\perp = A^\perp \otimes B^\perp \qquad (A \lhd B)^\perp = A^\perp \lhd B^\perp \tag{3}$$

while the equivalence relation $\equiv$ on formulas is generated by the following equations:

$$\begin{array}{ccc} A \parr (B \parr C) \equiv (A \parr B) \parr C & A \parr \circ \equiv A & A \parr B \equiv B \parr A \\ A \lhd (B \lhd C) \equiv (A \lhd B) \lhd C & A \lhd \circ \equiv A \equiv \circ \lhd A & \\ A \otimes (B \otimes C) \equiv (A \otimes B) \otimes C & A \otimes \circ \equiv A & A \otimes B \equiv B \otimes A \\ \text{associativity} & \text{unit laws} & \text{commutativity} \end{array} \tag{4}$$

To define the proof system for BV, we need the following **inference rules**:

$$\equiv \frac{B}{A} \text{ if } A \equiv B \qquad \mathsf{i}{\downarrow} \frac{\circ}{a \parr a^\perp} \qquad \mathsf{s} \frac{A \otimes (B \parr C)}{(A \otimes B) \parr C} \qquad \mathsf{q}{\downarrow} \frac{(A \lhd C) \parr (B \lhd D)}{(A \parr B) \lhd (C \parr D)} \tag{5}$$

equivalence **atomic interaction** **switch** **q-down**

where the rule $\mathsf{i}{\downarrow}$ represents an atomic axiom, the rule s is the standard weak distributivity of $\otimes$ over $\parr$, and the rule $\mathsf{q}{\downarrow}$ is the key rule that allows to express the interleaving law mentioned above.

A **derivation** $\Phi = \begin{array}{c} B \\ \Phi \| \\ A \end{array}$ with premise $B = \mathsf{pr}(\Phi)$ and conclusion $A = \mathsf{cn}(\Phi)$ is inductively defined from formulas using not only the rules in Equation (5), but also the binary connectives $\parr$, $\lhd$, and $\otimes$:

- Each formula A is a derivation A with premise A and conclusion A;
- if Φ and Ψ are derivations, then $\Phi \odot \Psi$ is a derivation with premise $\mathsf{pr}(\Phi) \odot \mathsf{pr}(\Psi)$ and conclusion $\mathsf{cn}(\Phi) \odot \mathsf{cn}(\Psi)$ for each $\odot \in \{\parr, \lhd, \otimes\}$;
- if Φ and Ψ are derivations, and ρ is an inference rule with premise $\mathsf{cn}(\Phi)$ and conclusion $\mathsf{pr}(\Psi)$, then $\rho \dfrac{\Phi}{\Psi}$ is a derivation with premise $\mathsf{pr}(\Phi)$ and conclusion $\mathsf{cn}(\Psi)$.

Remark 1. *Deep inference derivations are constructed using a more general structure than trees-like structure of sequent calculus. They can be seen as sequences of local rewriting rules applied to formulas, but they can also be seen as a generalization of the tree-like structure of sequent calculus, where the connectives are used to compose not only formulas, but also proofs.*

We write $\vdash_{\mathsf{BV}} A$ if there is a derivation $\Phi = \begin{array}{c} \circ \\ \Phi \| \\ A \end{array}$ with premise $\circ$ and conclusion A, and we call BV the logic defined as the set of formulas A such that $\vdash_{\mathsf{BV}} A$.

To conclude this section we briefly recall some results about the proof theory of BV, and the references where they were proved:

- **deduction theorem**: if $\vdash_{\mathsf{BV}} A \multimap B$, then there is a derivation Φ with premise A and conclusion B;
- **'cut'-admissibility** [10]: the rule $\mathsf{i}\uparrow \dfrac{A \otimes A^{\perp}}{\circ}$ is admissible in BV;
- **transitivity of the implication**: if $\vdash_{\mathsf{BV}} A \multimap B$ and $\vdash_{\mathsf{BV}} B \multimap C$, then $\vdash_{\mathsf{BV}} A \multimap C$;
- **the need for deep inference** [15]: BV does not admit a sound and complete cut-free sequent calculus;
- **completeness with respect to series-parallel pomsets** [7]: every series-parallel pomset $\mathcal{P}$ can be represented as a formula $A_{\mathcal{P}}$ in BV. Moreover, if $\mathcal{P}_1, \mathcal{P}_2$ are series-parallel pomsets such that $\mathcal{P}_1$ is a refinement of $\mathcal{P}_2$, then $\vdash_{\mathsf{BV}} A_{\mathcal{P}_1} \multimap A_{\mathcal{P}_2}$.

3 Extensions and Applications

We list here some extensions and applications of BV that have been developed in the last years, and that will be presented in the talk.

3.1 A logical account of process calculi

In [9], Bruscoli showed how one could use the connective $\triangleleft$ to give a logical account of the prefixing operator of process calculi, and how processes executions could be represented as proofs in BV providing a derivation-as-computation interpretation of proof search in the style of logic programming. The works have then been extended in [11, 12] to include operators of choice and restriction, and to give a logical account of the π-calculus (name passing).

In recent works [3, 1], we have been using this same idea to give a logical account of the process of extraction in the context of choreography languages. In particular, using the same idea of having a dedicated connective to represent the sequential composition of processes, we have taken inspiration from the logic BV to design the logic PiL where a non-associative version of the $\triangleleft$ is used to represent the sequential composition of processes, and where the other connectives and quantifiers (including nominal ones) are used to represent the other basic operators of choreographic languages and the π-calculus.

3.2 Categorical models of quantum computation

In [8], the authors provided a first categorical model of BV in the category of probabilistic coherence spaces, and define the notion of BV-category as a generalization of the notion of $*$-autonomous category, which is the standard categorical model of linear logic. As reformulated in [14], the definition of BV-category (with negation) is the one of a $*$-autonomous category $(\mathbf{C}, I, \otimes, \multimap)$ with an additional monoidal structure $(\mathbf{C}, J, \triangleleft)$, such that the two monoidal structures $(\mathbf{C}, I, \otimes)$ and $(\mathbf{C}, J, \triangleleft)$ form a **normal duoidal structure** in the sense of [6], that is, they are related by a natural transformation $w : (A \triangleleft B) \otimes (C \triangleleft D) \to (A \otimes C) \triangleleft (B \otimes D)$ that satisfies some coherence conditions (see also appendix of [5] for details).

In [5], we have provided a stronger definition of BV-category ensuring the presence of some additional natural transformations that are needed to give a categorical account of the 'cut-elimination' in BV, which were lacking in the original definition of [8]. For this purpose, we define a notion of **atomic flows** for BV to get a more direct account of the normalization process in BV and a notion of proof equivalence which were lacking in the literature.

$$\begin{array}{rl} \text{Formulas} & A, B ::= a \mid \circ \mid A \otimes B \mid A \multimap B \mid A \triangleleft B \qquad a \in \mathbb{A} \\ \text{Positive contexts} & \mathcal{P} ::= [\cdot] \mid \mathcal{P} \otimes A \mid A \otimes \mathcal{P} \mid A \multimap \mathcal{P} \mid \mathcal{N} \multimap A \mid \mathcal{P} \triangleleft A \mid A \triangleleft \mathcal{P} \\ \text{Negative contexts} & \mathcal{N} ::= \quad \mathcal{N} \otimes A \mid A \otimes \mathcal{N} \mid A \multimap \mathcal{N} \mid \mathcal{P} \multimap A \mid \mathcal{N} \triangleleft A \mid A \triangleleft \mathcal{N} \end{array}$$

$$\mathsf{ai}^{\circ}_{\downarrow}\ \frac{}{a \multimap a}\circ \qquad \mathsf{q}^{\circ}_{\downarrow}\ \frac{(A \multimap B) \triangleleft (C \multimap D)}{(A \triangleleft C) \multimap (B \triangleleft D)}\circ \qquad \mathsf{q}^{\bullet}_{\downarrow}\ \frac{(A \otimes B) \triangleleft (C \otimes D)}{(A \triangleleft C) \otimes (B \triangleleft D)}\bullet$$

$$\mathsf{s}^{\circ}_{\mathsf{L}}\ \frac{A \otimes (B \multimap C)}{(A \multimap B) \multimap C}\circ \qquad \mathsf{s}^{\circ}_{\mathsf{R}}\ \frac{(A \multimap B) \otimes C}{A \multimap (B \otimes C)}\circ \qquad \mathsf{s}^{\bullet}_{\mathsf{L}}\ \frac{(A \multimap B) \multimap C}{A \otimes (B \multimap C)}\bullet \qquad \mathsf{s}^{\bullet}_{\mathsf{R}}\ \frac{A \multimap (B \otimes C)}{(A \multimap B) \otimes C}\bullet$$

$$\mathsf{com}^{\otimes}\ \frac{A \otimes B}{B \otimes A} \qquad \mathsf{asso}^{\otimes}\ \frac{(A \otimes B) \otimes C}{A \otimes (B \otimes C)} \qquad \mathsf{asso}^{\triangleleft}_{\mathsf{L}}\ \frac{(A \triangleleft B) \triangleleft C}{A \triangleleft (B \triangleleft C)} \qquad \mathsf{asso}^{\triangleleft}_{\mathsf{R}}\ \frac{A \triangleleft (B \triangleleft C)}{(A \triangleleft B) \triangleleft C}$$

$$\mathsf{u}^{\otimes}_{\downarrow}\ \frac{A}{\circ \otimes A} \qquad \mathsf{u}^{\multimap}_{\downarrow}\ \frac{A}{\circ \multimap A} \qquad \mathsf{u}^{\triangleleft}_{\downarrow}\ \frac{A}{\circ \triangleleft A}\circ \qquad \mathsf{u}^{\triangleright}_{\downarrow}\ \frac{A}{A \triangleleft \circ}\circ \qquad \mathsf{cur}\ \frac{(A \otimes B) \multimap C}{A \multimap (B \multimap C)} \qquad \mathsf{ruc}\ \frac{A \multimap (B \multimap C)}{(A \otimes B) \multimap C}$$

Figure 1: Formulas, contexts, and inference rules of IBV. Rules with$\circ$ are applied in a positive context, while rules with$\bullet$ are applied in a negative context, and the other (dotted) rules can be applied in both contexts. The logic IBV is the set of formulas A such that there is a derivation of A from $\circ$ in IBV.

3.3 Type theories for imperative programming languages

We have been exploring the possibility of defining an intuitionistic version of BV that could be used as a type system for imperative programming languages, in the same way as intuitionistic logic is used as a type system for functional programming languages. The key idea is to enrich the type system based on intuitionistic logic with a non-commutative connective that allows to represent the sequential composition of commands.

In [4], we have defined IBV (see Figure 1), an intuitionistic version of BV, and we have shown that it satisfies desirable properties for a logic to be used as a type system for imperative programming languages, such as the deduction theorem and the transitivity of the sequent calculus. For this purpose, to deal with the problem of designing a deep inference system in an intuitionistic settings, we had to use positive and negative contexts to represent when a rule is applied in a co-variant or contravariant way.

However, also here the process of cut-elimination in deep inference systems makes it difficult to immediately see a Curry-Howard correspondence between the normalization process and the execution of programs, and we are currently working on a more direct presentation of the logic IBV that could make this correspondence more clear.

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Refutation systems for substructural logics

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A very important characteristic of logic is the ability, given a formula, to determine whether this formula is a theorem, and often the logics are given by deductive systems which allow us to prove the theorems. But the deductive system alone does not always give an answer: we will find a proof provided it exists. To obtain the negative results we either use semantics, some kind of models which we can use to refute the formula, or syntactic-like means like tableaux, or specially designed deductive systems which allow to establish the absence of proof by trying "to go backwards": starting with the formula that we want to prove and analyzing from which formulas it can be derived, and whether we are able to go all way back to the axioms. A notable example of this kind of refutation systems is Kleene's system $G3$:

> "The steps in the procedure will consist in listing the choices of the premise(s) for the inference of a given conclusion. In doing this, it is tedious to have to distinguish all the ways of applying the structural rules TCI. Therefore, for use in our version of Gentzen's decision procedure, we shall introduce a new Gentzen-type system G3...". ([5, p. 480])

In 1921, Łukasiewicz suggested (see [7]) to extend the regular deductive systems by adding the ability to reject the propositions:

> "I do not define these terms, and by *assertion* and *rejection* I mean the ways of behaviour with respect to the logical values, the ways known to everyone from his own experience. I wish to assert truth and only truth, and to reject falsehood and only falsehood." ([6, p. 91])

and in [3], Łukasiewicz introduced a formal system for rejection. Since then, for many different propositional, modal and relevant logics the Łukasiewicz-style rejection systems were constructed (for more information on these systems see [2]).

To obtain a refutation system for classical propositional logic, Łukasiewicz associated with each proposition (formula) A two meta-statements: A^+, meaning that A is accepted (a positive statement), and A^-, meaning that A is rejected (a negative statement). He then took a classical propositional calculus, transformed all axioms into positive statements, and converted the rules of modus ponens and substitution into the following inference rules:

$$\mathsf{MP} = \frac{\alpha^+,\ (\alpha \to \beta)^+}{\beta^+} \qquad \text{and} \qquad \mathsf{Sb} = \frac{\alpha^+}{\sigma(\alpha)^+}, \text{for any substitution } \sigma.$$

He further added an anti-axiom p^- (where p is a propositional variable) and introduced two new rules:

$$\mathsf{MT} = \frac{(\alpha \to \beta)^+,\ \beta^-}{\alpha^-} \qquad \text{and} \qquad \mathsf{RSb} = \frac{\sigma(\alpha)^-}{\alpha^-}, \text{for any substitution } \sigma.$$

He proved that, in this derivation system, the derivable positive statements are precisely the tautologies (that is, the theorems of classical logic), while the derivable negative statements are

precisely the non-tautologies. Hence, the resulting system is a complete refutation system for classical logic.

There are two different directions for extending Łukasiewicz's approach to a broader class of logics. Given a propositional calculus (PC), one can transform its axioms and rules into positive statements and inference rules for positive statements, and then add a single anti-axiom p^- together with new derivation rules (possibly infinitely many) for negative statements. This approach has proved to be very fruitful (see, e.g., [2]) and has yielded, for instance, a complete refutation system for intuitionistic logic (see [8]).

An alternative approach is to add only the two rules MT and RSb, but allow many (possibly infinitely many) anti-axioms (see, e.g., [1]). For example, after transforming the axioms of the intuitionistic propositional calculus into positive statements, one can add, as anti-axioms, those corresponding to each Jaśkowski algebra, and then add the rules MP, Sb, MT, and RSb. In this way, one obtains a complete refutation system for intuitionistic logic.

The latter approach is based on use of the Jankov characteristic formulas associated with counter-models – the Jaśkowski algebras. The characteristic formulas of algebra $\mathbf{A}$ posses a very important property: for every formula A refuted in $\mathbf{A}$, the characteristic formula of $\mathbf{A}$ can be derived from A which potentially allows one to use MT.

Our aim is to extend the latter approach to equational logics. The talk will cover as many of the following topics as possible, given the time restrictions:

- introducing a very general notion of refutation system needed to define refutation systems in the algebraic settings;
- introducing the notions of characteristic equations, which are the algebraic counterparts of the Jankov characteristic formulas [4], and algebras, and studying their properties;
- showing that every variety with the congruence extension property (CEP) generated by a class of characteristic algebras has a complete and sound refutation system;
- showing that in varieties with equationally definable compact congruences every finitely presented subdirectly irreducible algebra is characteristic;
- applying the results to the construction of refutation systems for a broad class of substructural logics, thus extending the results of [1] to a much broader class.

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On the existence of higher measures

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Abstract

Generalized descriptive set theory has its origins in the second half of the twentieth century, with early contributions such as those of Marshall H. Stone and later developments by Alan Mekler and Jouko Väänänen. In the last decade, the subject has undergone rapid expansion, with many central notions and structural results from classical descriptive set theory successfully extended to uncountable settings. This includes, for example, the study of reducibility of Borel equivalence relations on generalized Baire spaces, as well as extensions of classical dichotomy theorems and the theory of Souslin quasi-orders. More recently, the framework has been further generalized beyond the standard spaces ${}^{\kappa}\kappa$, incorporating more flexible topological structures and generalized metric notions, and even extending to cardinals λ satisfying $2^{<\lambda} = \lambda$, including singular cardinals.

Alongside these developments, several regularity properties, such as the Baire property, the perfect set property, and the Hurewicz dichotomy, have been systematically investigated in the generalized context. These results show that, at least consistently in certain cases, these properties can exhibit remarkably good behavior in the generalized context as well.

However, measure theory, a major topic of classical descriptive set theory with many applications to other areas of mathematics, has yet to find a proper analogue in the generalized setting. While there have been efforts to generalize concepts related to measures, such as the ideal of null sets, to the generalized Baire space, further progress has faced significant technical and conceptual obstacles. As a result, the question of whether a satisfactory notion of "measure" can be developed in this context has remained open.

In this talk I will present recent progress on this problem, based on joint work with Fernando Barrera and Vincenzo Dimonte, in which we resolve the question.

We show that, under very mild assumptions and for sufficiently large cardinals λ, it is impossible to define non-trivial generalized measures on λ^+-Borel spaces. In particular, no such measure exists on the generalized Baire space.

One of the main challenges in obtaining a meaningful impossibility theorem is to work with a class of "generalized measures" broad enough to encompass all reasonable generalizations of measure one might consider. We achieve this by relaxing the assumptions of classical measures as much as possible, introducing an "umbrella definition" that captures most (if not all) natural generalizations of measure to subspaces of ${}^{\kappa}\lambda$.

We allow our "generalized measures" to take values in an arbitrary positively totally ordered monoid equipped with an infinitary summation operation, and we work with a weakening of κ^+-algebras that is only required to be closed under unions (but not under complements or intersections), and to contain all open sets and singletons.

Within this general framework, we define λ^+-measures as functions satisfying natural analogues of monotonicity, additivity over families of size $< \lambda^+$, and a point-regularity condition that reflects the interaction between the measure and the topology. Despite the apparent flexibility of this definition, our main result shows that it is already too restrictive to admit non-trivial examples in the uncountable setting.

Our principal theorem establishes that if X is a subspace of weight $\lambda \geq \mathfrak{c}$ of some space of the form ${}^{\gamma}A$ for some set A and ordinal γ, equipped with the bounded topology, then there is no continuous weak λ^+-measure on X. In particular, assuming $\lambda^{<\kappa} = \lambda$ and

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$\lambda \geq \mathfrak{c}$, there is no continuous λ^+-measure on the generalized Baire space ${}^{\kappa}\lambda$. This result shows a fundamental obstruction to extending measure theory to generalized descriptive set theory: even under extremely permissive assumptions, non-trivial continuous measures cannot exist.

This is joint work with Fernando Barrera and Vincenzo Dimonte.

First-order Intuitionistic Multiplicative Linear Logic with Hyperdoctrines and Hilbert's Epsilon

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Abstract

We propose an IMELL doctrine whose base category is the full subcategory of Set consisting of finite powers of a given set U, using a particular monoidal poset M^ω with a non-trivial comonad. This doctrine is shown to be an ϵ-doctrine, and thus provides a categorical semantics for IMELL ϵ-calculus, a first order intuitionistic linear logic.

Introduction Girard's Linear Logic [4] has established itself as an exceptionally precise framework for the fine-grained analysis of classical logic and, most notably, intuitionistic logic. Beyond its logical significance, it has also proved highly fruitful as a logic of computation. In particular, it has played a central role in the study of cut-elimination and β-reduction, most famously through the demonstration that Lamping's algorithm realises maximal sharing,[7] as well as through the development of Geometry of Interaction [5] and has provided powerful tools for the modelling of concurrent processes, as recently exemplified in the work by Manara [12].

From a logical perspective, however, Linear Logic has largely left aside the systematic study of first order quantification — as in typed lambda calculus, second order quantification has been more successful [5]. Standard first-order quantifiers can of course be incorporated into Linear Logic via the usual rules, or even in proof nets as studied in the works of Heijltjes, Hughes and Straßburger [8]. However, while this extension is technically well behaved, it remains conceptually unsatisfactory. One would expect Linear Logic to articulate a genuinely linear notion of quantification, one that is not merely inherited from classical or intuitionistic logic, but instead reflects the distinctive resource-sensitive character of the theory itself. Indeed, usual quantifiers exhibit an irreducibly additive dimension, even within an otherwise purely multiplicative fragment, a phenomenon already emphasised, for instance, in the work of Fleury-Donnadieu and Quatrini on first-order Ludics [3].

One possible line of approach consists in treating individuals as propositions—understood as individual concepts—while propositions themselves are viewed as classes or terms, and in exploiting second-order quantification accordingly. This perspective is closely related to Girard's proposal of a transcendental syntax, in which quantification is internalised via higher-order structure rather than expressed through explicit additive operators [6].

A more down-to-earth alternative is to turn instead to Hilbert's subnector, ϵ [9, 1], with $\phi[\epsilon_x\, \phi[x]]$ expressing $\exists x\ \phi[x]$. Since an expression of the form $\epsilon_x\, \phi[x]$ is a term that effects quantification without appealing to additive connectives, it offers a promising route for reconciling quantification with the multiplicative discipline of Linear Logic. At the same time, we aim to retain truth-valued semantics, so as to preserve a meaningful notion of first-order model. The present work, which remains in progress, is informed by categorical logic for intuitionistic systems, and in particular by the theory of hyperdoctrines [10, 15], which provides a natural semantic environment for such investigations, pursuing the work of [2], on doctrines for linear logic and of [14, 13, 11] on epsilon in doctrines.

First-order IMELL with Epsilon The language of Intuitionistic Multiplicative Exponential Linear Logic (IMELL) + ϵ is the usual language of IMELL, extended with the operator ϵ forming terms t from formulae $\mathcal{F}$:

$$t := x \mid c \mid f(t, ..., t) \mid \epsilon_x \mathcal{F}$$

$$\mathcal{F} := \mathbf{1} \mid \bot \mid P(t, ..., t) \mid \mathcal{F} \otimes \mathcal{F} \mid \mathcal{F} \multimap \mathcal{F} \mid !\mathcal{F}$$

The meaning of $\epsilon_x \phi$ is, informally, "some x for which ϕ holds, if there are any". This is formally expressed by the corresponding sequent calculus rules, which are the usual rules of IMELL (see [4]), plus the ϵ-rules:

$$\varepsilon_L \frac{\Gamma, A(x) \vdash C}{\Gamma, A(\varepsilon_x A) \vdash C}\, x \notin free(\Gamma) \qquad \varepsilon_R \frac{\Gamma \vdash A(t)}{\Gamma \vdash A(\varepsilon_x A)}$$

First-order Hyperdoctrines with Epsilon Since usual (classical) models of the ϵ-calculus are not adequate for Linear Logic, we instead propose categorical semantics using hyperdoctrines, in the spirit of [13, 14].

Definition 1 (**Hyperdoctrine**). *A* hyperdoctrine*, or simply doctrine, is a functor $P : \boldsymbol{C}^{op} \to \boldsymbol{Pos}$, where $\boldsymbol{C}$ is a category with finite products (called the base category of P), and where $\boldsymbol{Pos}$ denotes the category of posets and monotone functions.*

The category $\mathbf{C}$ is to be seen as a category of contexts, while the poset $P(A)$ is understood as the collection of formulas over the context A, ordered by entailment. Typically, one requires that each $P(A)$ be equipped with certain operations, depending on the logic under consideration. In the case of IMELL, the relevant doctrines are those for which each $P(A)$ is an ordered monoid which, when viewed as a category, is monoidal closed and equipped with a comonad $!_A : P(A) \to P(A)$ satisfying $!(a) \leq !(a) \otimes !(a)$ and $!(a) \leq 1$ [2].

Definition 2 (**Syntactic Doctrine**). *Let T be an IMELL-theory over a language L. The* syntactic doctrine of T *is the doctrine $\mathsf{LT}_T : \boldsymbol{Ctx}_L^{op} \to \boldsymbol{Pos}$, where $\boldsymbol{Ctx}_L^{op}$ is the category of L-contexts, in which objects are finite lists of distinct variables, morphisms from $(x_1, \ldots, x_n)$ to $(y_1, \ldots, y_k)$ are lists of L-terms $(t_1, \ldots, t_k)$ whose free variables are $(x_1, \ldots, x_n)$, and for all $(x_1, \ldots, x_n)$, the poset $\mathsf{LT}_T(x_1, \ldots, x_n)$ is the Lindenbaum–Tarski algebra of T-equiprovable formulas of L whose free variables occur in $(x_1, \ldots, x_n)$.*

Binary products in $\mathbf{Ctx}_L^{op}$ are given by context concatenation, and the terminal object is the empty context. The action of LT_T on the morphisms of $\mathbf{Ctx}_L$ is defined by simultaneous substitution, as depicted in the following diagram.

$$\begin{array}{ccc} (x_1, ..., x_n) & \mathsf{LT}_T(x_1, ..., x_n) & \phi[t_1/y_1][t_2/y_2]...[t_k/y_k] \\ \Big\downarrow {\scriptstyle (t_1,...,t_k)} & \Big\uparrow {\scriptstyle \mathsf{LT}_T(t_1,...,t_k)} & \Big\uparrow \\ (y_1, ..., y_k) & \mathsf{LT}_T(y_1, ..., y_k) & \phi \end{array}$$

The monoidal operations and the comonad on each $\mathsf{LT}_T(x_1, ..., x_n)$ are given by the corresponding operations of linear logic.

Definition 3 (ϵ-**doctrine**). *An* ϵ-doctrine *is a doctrine $P : \boldsymbol{C}^{op} \to \boldsymbol{Pos}$ such that, for every pair of objects X, Y of $\boldsymbol{C}$ and every element ϕ of $P(X \times Y)$, there exists an arrow $\epsilon_\phi : X \to Y$ satisfying $\exists y\ \phi \leq P(\langle id_X, \epsilon_\phi \rangle)(\phi)$.*

Here $\exists y\ \phi$ is a simpler notation for $\Sigma(\pi_{X\times Y\mapsto X})(\phi)$ that is the left adjunct to the image by P of the projection $X \times Y \mapsto X$ in the base category. Accordingly, the class of doctrines associated with the IMELL calculus equipped with the ϵ subnector is precisely the class of IMELL ϵ-doctrines.

Categorical Semantics for IMELL + ϵ We propose to build for any non-empty set U an ϵ-doctrine P_U whose base category is the full subcategory of **Set** consisting of finite powers of U. We construct the adequate IMELL structure of the posets $P_U(X)$ in the following way.

Definition 4. *Let* $M = \{0\} \cup \{\frac{1}{2^n} | n \in \mathbb{N}\} \subset \mathbb{Q}$. *We define a* MLL phase structure *as the ordered monoid* $(M, \cdot, \leq)$ *where* $\cdot, \leq$ *are the usual multiplication and order on* $\mathbb{Q}$.

We define the operation $\multimap$ on M as $u \multimap v = \min(1, v/u)$ when $u \neq 0$ while $0 \multimap v = 1$. MLL formulae are interpreted as expected: $[\![\mathbf{1}]\!] = 1$, $[\![\bot]\!] = 0$, $[\![A \otimes B]\!] = [\![A]\!] \cdot [\![B]\!]$ and $[\![A \multimap B]\!] = [\![B]\!] \multimap [\![A]\!]$. It is easily checked that $(a \otimes b) \leq c$ iff $a \leq (b \multimap c)$.

The exponential ! is to be interpreted by a comonad, with: $[\![!A]\!] \leq [\![1]\!]$ (this one holds for any possible value of $[\![!A]\!]$), $[\![!A]\!] \leq [\![A]\!]$, $[\![!A]\!] \leq [\![!A \otimes !A]\!] = [\![!A]\!] \cdot [\![!A]\!]$.

In the structure M, this leaves only two possible values for $[\![!A]\!]$: 0 or 1, and consequently the interpretation of !-formulas is defined by $[\![!A]\!] = 1$ iff $[\![A]\!] = 1$ and otherwise $[\![!A]\!] = 0$.

Since this interpretation of ! would be too trivial, we use as phase structure ω copies of a MLL structure.

Definition 5. *We define an* IMELL phase structure *as* $(M^\omega, \cdot^\omega, \leq^\omega, !)$, *where* M *is a MLL structure, and the operations on* M^ω *are defined pointwise:*

$(u_n)_{n\in\mathbb{N}} \cdot^\omega (v_n)_{n\in\mathbb{N}} = (u_n \cdot v_n)_{n\in\mathbb{N}}$
$(u_n)_{n\in\mathbb{N}} \multimap^\omega (v_n)_{n\in\mathbb{N}} = (u_n \multimap v_n)_{n\in\mathbb{N}}$
$!(u_n)_{n\in\mathbb{N}} = (!u_n)_{n\in\mathbb{N}}$.
$(u_n)_{n\in\mathbb{N}} \leq^\omega (v_n)_{n\in\mathbb{N}}$ *when* $\forall n \in \mathbb{N}\ u_n \leq v_n$

IMELL formulae are interpreted as before, but with the operations on M^ω.

Interpretations of an IMELL-theory T are given by morphisms of doctrines: a morphism from P to P' is a pair $\langle F, f\rangle$, where $F : \mathbf{C} \to \mathbf{C}'$ is a functor between the respective base categories preserving finite products, and f is a natural transformation. If P, P' are IMELL ϵ-doctrines, then morphisms preserve the operations $\otimes, \multimap, !$ and the units.

Definition 6. *An* interpretation of an IMELL theory T *in* P *is a morphism* $\langle I, i\rangle : \mathcal{LT}_T \to P$.

Therefore, every IMELL ϵ-doctrine P_U as constructed above provides an interpretation of IMELL + ϵ. By construction, the soundness of this semantics for IMELL follows.

Proposition 1. *If an IMELL* + ϵ *formula is provable in* T, *then it is satisfied by every interpretation of* T.

Conclusion An intriguing question concerns τ, the subnector such that $\phi[\tau_x\phi]$ expresses $\forall x\, \phi[x]$. In a classical setting, both operators are not required, since $\tau_x\phi(x) = \epsilon_x\neg\phi(x)$. The same identification is possible in certain intermediate logics, namely intuitionistic logic enriched with additional axioms. How, then, could we define τ, the dual of ϵ, within our intuitionistic linear logic equipped with ϵ? In intuitionistic doctrines, universal quantification $\forall$ is available as an adjunction to projection, just as existential quantification $\exists$ is. So we can define τ as we defined ϵ.

Why, then, did we not introduce τ explicitly? The reason is that, with the ordered monoid we have chosen, we can accommodate either ϵ or τ, *but not both*. Indeed, every supremum is a maximum, that is, it is attained by at least one value. This guarantees the existence of the existential adjoint. However, it is not the case that every infimum is a minimum. For example, for the set $\{1/2^n \mid n \in \mathbb{N}\}$, the infimum is zero, which does not belong to the set.

To support both operators, one would need an ordered monoid equipped with a comonad satisfying this property in both directions. A finite ordered monoid would be a possible candidate, but to our knowledge no structure with all the required properties is currently known.

Since we do have $\bot$, we can define $\phi^\bot$, the linear negation of ϕ, as $\phi \multimap \bot$, and then define $\tau_x\phi(x)$ by $\epsilon_x\neg\phi(x) = \epsilon_x(\phi(x) \multimap \bot)$. This possibility deserves further investigation. However, unless χ is $\bot$, $\chi^\bot$ always collapses to $\bot$, so this approach does not yield a sufficiently fine-grained interpretation.

This issue is therefore left for further research.

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Ramsey's Witnesses

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Partition regularity is one of the key notions in combinatorics. In general, a family $\mathcal{F}$ of subsets of a set X is *partition regular* (PR) over X if it is closed under taking supersets and for all finite colourings $X = C_1 \cup \ldots \cup C_r$ there exists $i \leq r$ such that $C_i \in \mathcal{F}$; and it is *strongly partition regular* (SPR) over X if, furthermore, for all $Y \in \mathcal{F}$ the family $\mathcal{F} \cap \mathcal{P}(Y)$ is PR over Y, i.e. if for all finite colourings $Y = C_1 \cup \ldots \cup C_r$ there exists $i \leq r$ such that $C_i \in \mathcal{F}$.

Arithmetic Ramsey theory is the area of mathematics that studies which family of subsets of $\mathbb{N}$, defined in terms of arithmetic operations, are PR. Problems in this area have been tackled by employing a variety of methods; in recent years, those based on the strict connection between spaces of ultrafilters and their algebra and nonstandard extensions of $\mathbb{N}$ have been extensively studied.

Elements of some nonstandard extension ${}^*\mathbb{N}$ are called hypernatural numbers, and they and their tuples bear a natural relation to ultrafilters on $\mathbb{N}$ and its Cartesian powers. Namely, for $(\alpha_1, \ldots, \alpha_k) \in {}^*\mathbb{N}^k$, the set

$$u_{(\alpha_1,\ldots,\alpha_k)} \coloneqq \{A \subseteq \mathbb{N}^k \mid (\alpha_1, \ldots, \alpha_k) \in {}^*A\}$$

belongs to $\beta\mathbb{N}^k$—that is, it is an ultrafilter on $\mathbb{N}^k$. Conversely, every $u \in \beta\mathbb{N}^k$ is of the form $u = u_{(\alpha_1,\ldots,\alpha_k)}$ for a suitable $(\alpha_1, \ldots, \alpha_k) \in {}^*\mathbb{N}^k$, provided ${}^*\mathbb{N}$ is $\mathfrak{c}^+$-saturated.

The connection between partition regularity and ultrafilters stems from the well-known facts that

1. $\mathcal{F}$ is PR over X if and only if there exists an ultrafilter u over X such that $u \subseteq \mathcal{F}$; such a u is called a *witness* of the partition regularity of $\mathcal{F}$; and
2. $\mathcal{F}$ is SPR over X if and only if $\mathcal{F} \neq \emptyset$ and for all $Y \in \mathcal{F}$ there is a witness $u \subseteq \mathcal{F}$ containing Y.

In nonstandard terms, the relationship between PR and nonstandard extensions is restated using the following known characterization.

Theorem 1. A formula $\phi(x_1, \ldots, x_n)$ is PR, that is, for every finite colouring $\mathbb{N} = C_1 \cup \cdots \cup C_r$ there is a monochromatic $(a_1, \ldots, a_n)$ such that $\mathbb{N} \models \phi(a_1, \ldots, a_n)$, if and only if there are $\alpha_1 \sim \cdots \sim \alpha_n$ such that ${}^*\mathbb{N} \models \phi(\alpha_1, \ldots, \alpha_n)$.

Many statements in combinatorial number theory involve the existence of infinite sets satisfying certain arithmetic restriction. These can be uniformly restated in terms of the so-called Ramsey partition regularity.

Definition 2. We call an ultrafilter $u \in \beta\mathbb{N}^2$ a Ramsey's witness if for all $A \in u$ there exists an infinite $H \subseteq \mathbb{N}$ such that $[H]^2 \subseteq A$. We also write RW for the subset of $\beta\mathbb{N}^2$ consisting of all u as above, and say that $(\alpha, \beta) \in {}^*\mathbb{N}^2$ is a *Ramsey pair*, if $u_{(\alpha,\beta)} \in \mathrm{RW}$.

Roughly speaking, an ultrafilter is a Ramsey witness precisely when its elements are witnesses of Ramsey Theorem for pairs, i.e. they all contain infinite configurations of the form $[X]^2$, for $X \subseteq \mathbb{N}$ infinite.

Ramsey witnesses can be used to determine whether various configurations involving polynomials and exponentials are Ramsey partition regular over the natural number, providing negative answers to several questions recently posed by Kra, Moreira, Richter and Robertson in [2]. In particular, we will discuss how it is possible to procuce a (relatively simple) nonstandard proof of the following result, originally proven by Hindman in the '80s:

Theorem 3. Let $\mathcal{F}$ be the family of those $A \subseteq \mathbb{N}$ such that

$$\exists H \subseteq \mathbb{N} \ H \text{ is infinite and } \forall (x, y) \in [H]^2 \ (x + y, x \cdot y \in A).$$

Then $\mathcal{F}$ is not PR.

Our nonstandard characterization are easy enough to handle that they allows to generalize the above result in several different directions. The spirit of these generalizations is that Ramsey PR is rare. If time allows, we will show how rare it is by discussing the following two results.

Theorem 4. Let $P, Q \in \mathbb{Z}[x]$ be nonzero polynomials with $P(0) = Q(0) = 0$ and $a, n \in \mathbb{N}$. The following are equivalent.

1. There are $\alpha \sim \beta \sim \gamma$ with (α, β) a Ramsey pair such that $a\alpha^n + P(\beta) = Q(\gamma)$;
2. $ax^n + P(y) - Q(z)$ is either $a(x + y - z)$ or $a(x - y + z)$.

Theorem 5. If $\alpha \sim \beta \sim \gamma$ and (α, β) is a Ramsey pair, then α, β, γ do not form, in any order, an arithmetic progression.

This is a joint work with Mauro Di Nasso, Marcello Mamino, Rosario Mennuni, and Mariaclara Ragosta (see [1]).

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Projective curves and weak second-order logic

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This abstract is based on the preprint [1]. Given an algebraically closed field K of characteristic 0, we consider the poset $\mathrm{Var}(K)$ of irreducible proper Zariski-closed subsets of $\mathbb{P}^2(K)$ (points and curves), ordered by inclusion. In [3] Marcus Tressl asks whether the theory of this structure is undecidable and whether it interprets the integers. We give a positive answer to both questions, and we prove that the complete theory of the poset is sensitive to the transcendence degree of the field. To this aim we use a bi-interpretability result.

The poset $\mathrm{Var}(K)$ is interdefinable with the incidence relation between points and curves in $\mathbb{P}^2(K)$ (because there are no proper inclusions between irreducible curves). Given $n \in \mathbb{N}$, let $\mathrm{Var}_{\leq n}(K)$ be the substructure obtained by considering only curves of degree $\leq n$ and points. It is well known that $\mathrm{Var}_{\leq 1}(K)$ interprets K (after fixing some parameters to determine the projective frame), namely we can reconstruct the field from the incidence relation between points and lines. Using this fact it is not difficult to show that, for any fixed n, $\mathrm{Var}_{\leq n}(K)$ is bi-interpretable with K.

If we additionally assume that K is algebraically closed of characteristic zero, a result of [2] implies that $\mathrm{Var}_{\leq 2}(K)$ is a definable subset of $\mathrm{Var}(K)$. The cited result says that a curve C has degree ≤ 2 if and only if, for any point P on the curve, there is another curve that intersects C only at P. Since we already know that $\mathrm{Var}_{\leq 2}(K)$ interprets K, it follows that $\mathrm{Var}(K)$ interprets K as well, but it is easy to see that the converse fails. We show however that $\mathrm{Var}(K)$ is bi-interpretable with the two-sorted structure $(K, \mathrm{Fin}(K))$ obtained by adding to the field K a sort for its finite subsets, together with the membership relation between K and $\mathrm{Fin}(K)$. In this structure we can define the subring of integers $\mathbb{Z} \subset K$, so the theory is undecidable.

For K algebraically closed, of characteristic zero, and of infinite transcendence degree, we show that the complete theory of $(K, \mathrm{Fin}(K))$ has the form $T_{\mathrm{rec}} \cup T_{\mathbb{N}}$ where T_{rec} is a recursively axiomatized theory and $T_{\mathbb{N}}$ is the complete theory of $(\mathbb{N}, +, \cdot)$ relativized to the predicate $\mathbf{N}$ which defines $\mathbb{N} \subset K$ in $(K, \mathrm{Fin}(K))$. In particular $T_{\mathrm{rec}} \cup T_{\mathbb{N}}$ is the complete theory of $(\mathbb{C}, \mathrm{Fin}(\mathbb{C}))$.

From the axiomatization it follows that a structure of the form $(K, \mathrm{Fin}(K))$ is elementarily equivalent to $(\mathbb{C}, \mathrm{Fin}(\mathbb{C}))$ if and only if K is an algebraically closed field of characteristic zero and infinite transcendence degree. In particular, letting $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$, we have that $(\overline{\mathbb{Q}}, \mathrm{Fin}(\overline{\mathbb{Q}}))$ is not elementarily equivalent to $(\mathbb{C}, \mathrm{Fin}(\mathbb{C}))$. Thanks to the bi-interpretability result we then conclude that $\mathrm{Var}(\overline{\mathbb{Q}})$ is not elementarily equivalent to $\mathrm{Var}(\mathbb{C})$.

Since the notion of finite set is not expressible in first-order logic, there are non-standard models $\mathcal{K}$ of T_{rec} which are not of the form $(K, \mathrm{Fin}(K))$, and in which the predicate $\mathbf{N}$ defines a non-standard model of Peano Arithmetic. It is important to study these models because the completeness of $T_{\mathrm{rec}} \cup T_{\mathbb{N}}$ is obtained through a back and forth argument between saturated models with the same $\mathbf{N}$. This uses a definable version of Hilbert's basis theorem for non-standard polynomials based on a proof of Dickson's lemma which can be carried out in Peano Arithmetic. We also use the existence of generic zeros for definable prime ideals of non-standard

polynomials. All this implies that the type over $\mathbf{N}$ of any tuple of elements from a model of T_{rec} is determined by a (possibly non-standard) polynomial ideal.

A non-standard model of T_{rec} has the form $(K, \mathcal{F}(K))$ where $\mathcal{F}(K)$ is a *definable finite power set of* K and K is a field which satisfies a strengthening (depending on $\mathcal{F}(K)$) of the property of being algebraically closed, of infinite transcendence degree, and of characteristic zero. The elements of $\mathcal{F}(K)$ are codes for certain subsets of K which we call *hyperfinite*. We show that inside any such model $\mathcal{K} = (K, \mathcal{F}(K))$ we can define a model $\mathbf{N}(\mathcal{K}) \subset K$ of first-order Peano Arithmetic and that the complete theory of $(K, \mathcal{F}(K))$ is determined by T_{rec} and the complete theory of $\mathbf{N}(\mathcal{K})$. When $\mathcal{F}(K) = \mathrm{Fin}(K)$, we have $\mathbf{N}(\mathcal{K}) = \mathbb{N}$.

Finally, we show that $\mathbb{N}$ is *stably embedded* in $(\mathbb{C}, \mathrm{Fin}(\mathbb{C}))$ in the following strong sense: any subset of $\mathbb{N}^n$ definable with parameters in $(\mathbb{C}, \mathrm{Fin}(\mathbb{C}))$ is already definable in $(\mathbb{N}, +, \cdot)$, implying that there are at most countably many such subsets. By contrast, it is well known that, if $\mathbb{R}$ is the real field, then in $(\mathbb{R}, \mathbb{N})$ one can define, with parameters, every subset of $\mathbb{N}$.

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The logic of arithmetic Markov chains

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One of the key results in the theory of Riesz spaces is *Yosida duality* [7], which states that the category of metrically-complete, Archimedean and unital Riesz spaces is dual to the category of compact Hausdorff spaces. This theorem was part of a large body of results obtained during the 1930s and 1940s (see [4]).

We introduce arithmetic Markov chains and prove that their transition semantics is dual to modal ℓ-groups and hence equivalent, via Mundici's equivalence, to modal MV-algebras. This yields a modal Łukasiewicz logic for probabilistic transition systems without real scalar connectives, generalizing interesting variations of Yosida duality. One of these was introduced in [1], where the authors generalized Yosida duality to the category of *ℓ-groups*. To do so, they defined the notion of *arithmetic space* (see [1, Definition 2.2]), that is, a topological space X with a continuous function $\zeta\colon X \to \mathbb{N}$, where the set of naturals is endowed with the upper topology on the divisibility order. Typical examples of arithmetic spaces are $(\mathbb{R}, \mathsf{den})$, where $\mathsf{den}(p/q) = q$ and $\mathsf{den}(x) = 0$ if x is irrational, or $(\mathbb{R}^I, \mathsf{den})$, where $\mathsf{den}(\{\, x_i \,\}_{i\in I}) \coloneqq \mathsf{lcm}\,\{\, \mathsf{den}(x_i) \mid i \in I \,\}$. Arithmetic spaces form a category whose arrows are the so-called *a-maps*, i.e., continuous maps $f\colon (X, \zeta_X) \to (Y, \zeta_Y)$ such that $\zeta_Y(f(x))$ divides $\zeta_X(x)$ for all $x \in X$. The set of a-maps from X to Y is denoted by $C_a(X, Y)$. The main result is [1, Theorem 7.10] where the authors prove that the category of metrically-complete, Archimedean and unital ℓ-groups is dual to the full subcategory of *normal arithmetic spaces*, i.e. arithmetic spaces of type (K, den), where K is a compact subspace of $\mathbb{R}^I$ for some set I.

Another interesting variation of Yosida duality is the "stochastic" version. In [3] the authors defined *Markov chains* as coalgebras for a functor R on the category of compact Hausdorff spaces that send a space X into the space of linear and positive functionals of $C(X, \mathbb{R})$ such that $F(1_X) \leq 1$, where 1_X is the 1-constant function (see [3, Definition 2.3]). Moreover, in [3, Definition 4.1], they defined the variety of *modal Riesz spaces*. These structures are built endowing a pair (V, u) —where V is a Riesz space and u is a distinguished positive element— with a modality $\Diamond$, which is a positive and linear operator such that $\Diamond(u) \leq u$. Finally, they proved in [3, Theorem 5.1] a duality between the category of metrically-complete, Archimedean, unital modal Riesz spaces and the category of Markov chains. This paves the way for an algebraic study of Markov processes.

In this talk, we develop a modal logic for Markov chains which avoids the continuously many scalar connectives appearing in modal Riesz-space semantics using the notion of *modal ℓ-group* (see [5, Subsection 2.3]). A modal ℓ-group is an ℓ-group with a distinguished positive element and a modality $\Diamond$; the modality is defined analogously to the case of modal Riesz spaces. An *arithmetic Markov chain* is a coalgebra $\alpha\colon (X, \zeta) \to R_a(X, \zeta)$, where $R_a(X, \zeta)$ is the arithmetic space of positive additive subunital functionals on $C_a(X, \mathbb{R})$. The map ζ may be regarded as assigning an arithmetic weight to each state, constraining the admissible transition probabilities. We prove the following generalization of the duality of [3] and [1].

Theorem 1. *The category of metrically-complete, Archimedean, unital modal ℓ-groups is dual to the category of arithmetic Markov chains, where (X, ζ) is a normal arithmetic space.*

As a consequence of this theorem, each modal ℓ-group is isomorphic to a unique ℓ-group of type $C_\Diamond(X,\zeta) \coloneqq (C_a(X,\zeta),\Diamond)$ up to a homeomorphism of (X,ζ). Furthermore, we extend Mundici's equivalence (see [2, Section 7]) between unital ℓ-groups and MV-algebras to an equivalence $\Gamma_\Diamond$ between modal ℓ-groups and *modal MV-algebras*. As a corollary, we get the following result.

Corollary 2. *There exists an adjunction between the category of modal MV-algebras and the category of arithmetic Markov chains.*

This result suggests that Markov chains can be studied within the standard language of Łukasiewicz logic extended with the modal operator $\Diamond$. This observation naturally leads us to introduce the *Markov-Łukasiewicz logic*, axiomatized as follows.

Definition 3. The axioms and rules of the Markov-Łukasiewicz logic are:

- Axioms:

  (Ł) All axioms of Łukasiewicz logic,

  (MŁ1) $\Diamond(\bot) \to \bot$,

  (MŁ2) $\Diamond(\neg\varphi) \to \neg\Diamond(\varphi)$,

  (MŁ3) $\Diamond(\varphi \oplus \psi) \leftrightarrow (\Diamond(\varphi) \oplus \Diamond(\neg\varphi \wedge \psi))$.

- Rules:

  (MP) modus ponens: from φ and $\varphi \to \psi$ derive ψ,

  (M) from $\varphi \to \psi$ derive $\Diamond(\varphi) \to \Diamond(\psi)$.

The equivalent algebraic semantics of the Markov-Łukasiewicz logic is given by the class of *modal MV-algebras*. Furthermore, the logic has the following remarkable semantics.

Definition 4.

- A *Markov model* is a pair $\mathfrak{M} \coloneqq ((X,\alpha),e)$, where (X,α) is an arithmetic Markov chain and $e\colon \mathsf{Var} \times X \to [0,1]$ is a function such that for all $p \in \mathsf{Var}$ the function $f_p\colon X \to [0,1]$ given by $f_p(x) \coloneqq e(p,x)$ is an a-map.

- Let φ be a formula of Markov-Łukasiewicz logic. The *truth function of φ in $\mathfrak{M}$*, denoted by $t^{\mathfrak{M}}_\varphi\colon X \to \mathbb{R}$, is the a-map in $\Gamma_\Diamond(C_\Diamond(X,\alpha),1_X)$ defined inductively on the structure of φ, starting from the assignment $t^{\mathfrak{M}}_p(x) \coloneqq f_p(x)$ for $p \in \mathsf{Var}$ and $x \in X$.

- We say that a formula φ is *valid* at x in $\mathfrak{M}$ if $t^{\mathfrak{M}}_\varphi(x) = 1$. A formula is *valid* if it is valid at all points in all Markov models.

We show that Markov-Łukasiewicz logic together with an additional infinitary rule is complete with respect to the class of Markov models.

Theorem 5. *An arbitrary formula φ is valid if and only if φ is derivable in Markov-Łukasiewicz logic expanded by the* Archimedean *rule:*

(A) from $n\varphi \to \psi$ for all $n \in \mathbb{N}$ deduce $\neg\varphi$.

The completeness theorem also applies to the denominator-zero fragment of arithmetic Markov chains, which recovers the Markov processes considered in [3]. In this way, ordinary probabilistic transition system can be interpreted within a modal expansion of the standard language of Łukasiewicz logic, avoiding the use of real scalar connectives. The price to pay is the infinitary Archimedean rule. This rule is not needed in modal Riesz logic because free modal Riesz spaces are Archimedean (see [6, Theorem 1]). However, at the moment is open whether this rule is necessary in our framework, or equivalently whether the free modal MV-algebras are Archimedean.

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Defeasible Arithmetic: Non-monotonic Reasoning with Constraints and Assignments

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Abstract

We extend propositional defeasible logic, a non-monotonic framework for dealing with exceptions and conflicting rules, to numeric statements; in particular, the syntax is enriched with assignment and constraint literals.

In the context of non-monotonic reasoning [1], Defeasible Logic, DL, [8] turned out to be a useful and versatile tool. Throughout the years, many variants of DL have been proposed to model different application areas: agents [2, 3, 5], legal reasoning [4, 6], and business processes [9].

Defeasible Logic is a non-monotonic formal framework designed to handle incomplete or conflicting information. Unlike classical logic, where conclusions hold absolutely once derived, defeasible logic allows conclusions to be defeated — that is, withdrawn — when new, stronger evidence or rules come into play. It models the way humans reason in everyday life: drawing tentative conclusions that remain valid unless overridden by exceptions or contrary evidence.

More precisely, a (propositional) defeasible theory D is a tuple $(F, R, >)$, where F is the set of facts (indisputable statements), R is the set of rules, and $>$ is a binary relation over R. R is partitioned into three distinct subsets of rules, with different meanings to draw different "types of conclusions". *Strict rules*, represented by $\rightarrow$, are rules in the classical fashion: whenever the premises are the case, so is the conclusion. *Defeasible rules*, denoted by $\Rightarrow$, represent (along which defeaters) the non-monotonic part of the logic: if the premises are the case, then typically the conclusion holds, unless we have contrary evidence that prevents us to draw such a conclusion. Lastly, *defeaters* ($\rightsquigarrow$) are special rules whose purpose is to prevent contrary evidence to be the case. Every rule r, denoted by $r\colon A(r) \hookrightarrow C(r)$ with $\hookrightarrow \in \{\rightarrow, \Rightarrow, \rightsquigarrow\}$, has a set of antecedents $A(r)$, which is a set of literals, and a consequent $C(r)$, given by a single literal; e.g., the rule $v\colon a, b \Rightarrow c$ allows to derive c if a, b are true and every possible rule for $\neg c$ is defeated (by v itself or by another rule deriving c). It follows that in DL, through defeasible rules and defeaters, we can represent in a natural way exceptions (and exceptions to exceptions, and so forth), as well as conflicting rules.

Finally, there is the superiority relation $>$, which is a binary relation between rules whose objective is to solve conflicts. Given two rules r and s, if we have $(r, s) \in >$ (or

simply $r > s$), then in the scenario where both rules can be activated, r's conclusion will be preferred to s's.

As an example, let us consider the following defeasible theory D where $F = \{a, \neg b\}$, $>= \{s > u\}$, and R is given by the following four rules:

$$\begin{array}{rrcl} r\colon & a & \to & c, \\ s\colon & c & \Rightarrow & d, \\ t\colon & e & \Rightarrow & \neg d, \\ u\colon & \neg b & \rightsquigarrow & \neg d. \end{array}$$

Given the proof conditions of defeasible logic, since a is true (being a fact) and r is a strict rule, we can immediately derive that c holds. For the literal d the situation is less clear, because, in addition to rule s which allows to derive d, there are t and u having as consequent $\neg d$. To derive d, the reasoning is the following: 1) since e, the only antecedent of t, is not a fact and is not the consequent of a rule, this means that rule t is not applicable and is discarded; 2) although $\neg b$ is true (being a fact), and thus u is applicable, the superiority relation $s > u$ ensures that, in case of conflict, rule s prevails. In conclusion, we can apply s to derive d. Let us observe that, even if we remove rule s from the theory, the literal $\neg d$ can not still be derived; in fact, rule e remains inapplicable (since e is not derivable) and u, being a defeater, can only be used to block a derivation, but not to derive a literal.

Recently, we proposed an arithmetic extension of propositional defeasible logic [7]. More precisely, we add to the language six new elements, out of which two new types of literals and one new type of rules are formed. The fundamental components of the novel signature are *numeric variables*, *numeric constants*, *relational operators*, *operations*, *parentheses*, and the *assignment symbol*. For example, within the extended syntax we can express the following arithmetic defeasible theory that mixes together propositional and numeric variables:

$$\begin{array}{rrcl} r\colon & & \Rightarrow & a, \\ s\colon & a & \Rightarrow & x := 3, \\ t\colon & x \leq 4 & \Rightarrow & y := x + 2. \end{array}$$

For what concerns this last example, the plain extension to the arithmetic case of standard defeasible reasoning allows to derive that a holds, x is assigned the value 3, while to y is assigned the value 5.

Finally, beyond the standard inferential method of defeasible logic, we developed a new inferential principle that allows to validate numeric constraints even in absence of derived assignment literals for the related numeric variables. For example, although in the following theory no assignment for x can be derived, we are still able to infer a.

$$\begin{array}{rrcl} r'\colon & & \Rightarrow & x := 2, \\ s'\colon & & \Rightarrow & x := 3, \\ t'\colon & x \leq 4 & \Rightarrow & a. \end{array}$$

The idea is that if all possible assignments are evaluated and all possible values satisfy a constraint, then that constraint can be derived.

The goal of our arithmetic extension of DL is to furnish a non-monotonic framework to properly treat arithmetic in natural reasoning, as well as other in other contexts (e.g., formalization of tax or penal codes in legal reasoning); despite its usefulness, such a formal theoretical tool seems to be missing in the literature.

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Profinite Rigidity of Crystallographic Groups arising from Lie Theory

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Abstract

We prove that every finite direct product of crystallographic groups arising from an irreducible root system (in the sense of Lie theory) is profinitely rigid, and, equivalently, first-order rigid. This generalizes recent proofs of profinite rigidity for affine Coxeter groups [4, 13]. Our proof is essentially model-theoretic, leveraging on the classic equivalence, due to Oger [12], between elementary equivalence and isomorphism of profinite completions for finitely generated abelian-by-finite groups. The techniques we use combine integral representation theory, crystallography of Coxeter groups as developed in [8, 9], and Lie-theoretic considerations on root lattices.

1 Context and Motivation

A finitely generated residually finite group G is said to be *profinitely rigid* if, for any finitely generated residually finite group H, the isomorphism $\widehat{G} \cong \widehat{H}$ of profinite completions implies that $G \cong H$. In recent years, the problem of profinite rigidity of finitely generated groups has become central to group theory, driven by Remeslennikov's celebrated open question [10, Question 5.48], asking whether non-abelian free group are profinitely rigid. Within this framework, the problem of profinite rigidity of Coxeter groups has been considered in [6, 11], with [11] focusing specifically on affine Coxeter groups and posing the question of their profinite rigidity.

A landmark result of Oger [12] shows that for finitely generated abelian-by-finite groups (a class that includes all crystallographic groups) the problem of profinite rigidity is equivalent the one of *first-order rigidity*, which asks whether the given group is, up to isomorphism, the only finitely generated model of its first-order theory. This transforms an analysis of profinite group theory, into a model-theoretic one.

Building on this perspective, the profinite rigidity (and first-order rigidity) of affine Coxeter groups was established in [13] by using mathematical logic, with an independent group-theoretic proof given in [4]. The present work substantially extends [13], proving the rigidity of a much broader class of crystallographic groups arising from root systems and a Lie-theoretic context, including non-split extensions.

2 Overview and Main Results

A group G is *crystallographic* if it admits a short exact sequence $1 \to T \to G \to W_0 \to 1$ such that the subgroup T, called the *translation subgroup* of G, is isomorphic to $\mathbb{Z}^n$, and W_0, the *point group* of G, is finite and acts faithfully on T by conjugation. By a classic result of Zassenhaus [14], these are precisely the groups of isometries of an n-dimensional Euclidean real

vector space V whose action on V is discrete and cocompact. The action of W_0 on T induces a natural structure of $\mathbb{Z}[W_0]$-lattice L on T, said the *translation lattice* of G (cf. [3, 2.2, 2.5]).

In this work, we prove two major profinite rigidity results. The first one relies on abstract algebraic hypotheses, and techniques from model theory of groups and integral representation theory. First, we show that the translation subgroup T of a crystallographic group G is $\emptyset$-definable by the formula $\phi(x) \equiv \forall y\,([x, y^k] = e)$, where k is the order of the point group W_0 (cf. [3, 3.3]). Therefore, if G' is a finitely generated group elementary equivalent to G, the elementary equivalence transfers the whole extension structure, and we obtain that G' is itself crystallographic with translation subgroup $T' \cong T$, and point group $W_0' \cong W_0$, while the translation lattices L, L' belong to the same genus as $\mathbb{Z}[W_0]$-lattices (cf. [3, 3.4, 3.7]). We then prove that this genus equality can be lifted to a $\mathbb{Z}[W_0]$-lattice isomorphism under a strong assumption of irreducibility. The lattice L is said to be *absolutely irreducible* if its scalar extension $K \otimes_{\mathbb{Z}} L$ is irreducible as a $K[W_0]$-module, for every field extension $K/\mathbb{Q}$. By a theorem of Oger and Lasserre [7, Theorem 2.2], it is known that if G and G' are polycyclic-by-finite groups such that $G \equiv G'$, and G admits a decomposition $G \cong \prod_{i\in[1,m]} G_i$, then there exists also a decomposition $G' \cong \prod_{i\in[1,m]} G_i'$ such that $G_i \equiv G_i'$, for all $i \in [1, m]$. Consequently, since crystallographic groups are polycyclic-by-finite, we obtain the following result.

Theorem 1.1. *Finite direct products of absolutely irreducible split crystallographic groups are profinitely rigid (equiv. first-order rigid).*

The second main theorem applies to the class of crystallographic groups arising in *Lie theory*. Following [9, 8], we say that G *arises from an irreducible root system* if it admits an affine realization in some Euclidean space V whose point group W_0 is generated by reflections and essential (i.e., it has trivial fixed subspace $V^{W_0} = \{0\}$). In this case, W_0 is always the Weyl group of an irreducible root system R in V (cf. [3, 6.2, 6.3]). These groups, which are essentially all crystallographic groups compatible with the standard action of a Weyl group in Lie theory, were explicitly computed by Martinais in [8]. Extending a result of Bourbaki [1], we prove that if R is an irreducible root system, then G is absolutely irreducible (cf. [3, 5.27, 6.5]). Consequently, Theorem 1.1 applies, and the lattice structure of L is encoded in the first-order theory of G. Then, for every extension of W_0 by L computed by [8], we find a suitable first-order sentence in the language of groups characterizing it uniquely among all the possible extensions. These sentences not only detect the splitting of the extension, but further discriminate the non-split ones by specifying arithmetic properties of the cosets of the simple reflections. The argument requires a detailed case analysis for root systems types, and non-trivial properties of root systems and Weyl groups (cf. [3, 6.10, 6.16]). As a result, we obtain the following.

Theorem 1.2. *Finite direct products of crystallographic groups arising from an irreducible root system are profinitely rigid (equiv. first-order rigid).*

These theorems generalize the results in [4, 13] in various different directions. On the one hand, Theorem 1.1 simply assumes absolute irreducibility of the integral representation associated to the crystallographic group G, without asking for specific properties of any of its affine realizations. On the other hand, Theorem 1.2 requires that the group can be realized as the group of symmetries of a root system (as in the case of affine Coxeter groups), but it generalizes [4, 13] twofold: firstly, in considering *all* lattices compatible with Weyl group actions (not only the ones associated to the affine Coxeter groups), and, most importantly, in considering *any* group extension associated to any such lattice, not only the split ones.

Notice that for the problem of profinite rigidity *some* assumptions on the given crystallographic group are necessary, as e.g. for every $n \geq 85$ there exist split crystallographic groups

G_1, G_2 of dimension $\phi(n)$ such that $G_1 \ncong G_2$ but $\widehat{G}_1 \cong \widehat{G}_2$, where $\phi(n)$ is Euler's function (cf. [2]). In particular, for any $p \geq 23$, there are such groups of the form $\mathbb{Z}^{p-1} \rtimes \mathbb{Z}/p\mathbb{Z}$ (see [5, p. 204-205]). Similarly, there are known examples of non-isomorphic non-split crystallographic groups of the same genus (i.e., elementarily equivalent) which have isomorphic translation lattices (cf. [5, p. 205]).

3 Significance

Our results constitute the first profinite rigidity theorem for non-split crystallographic groups of arbitrary rank, settling a natural question left open by all previous work. The proof is model-theoretic in character: both main theorems are established by proving first-order rigidity, and the deduction of profinite rigidity follows from Oger's fundamental equivalence [12]. The approach combines four distinct mathematical threads: the model theory of abelian-by-finite groups, integral representation theory, crystallography of Coxeter groups (as developed in [8, 9]), and the Lie theory of root systems and Weyl groups in a way that we believe might be of independent interest for each of these communities.

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Towards Higher-Order Logarithmic Space

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Abstract

In this talk we introduce the type-2 analogue of FLOGSPACE, defined through oracle Turing machines, answering a question raised by Kawamura [4]. We show that this class, dubbed *Easy Feasible Functionals* (EFF) is robust, being closed by both type-0 and type-1 composition.

Higher-order computation was born together with computer science. We could say that the λ-calculus, the standard way to model higher-order computation, was the first universal theory of functions. However, when the study of computability theory shifted towards the study of complexity, the interest in higher-order computation diminished, being relegated mostly to the theory of programming languages. In fact, complexity theory is typically *first-order*, investigating functions of type $\mathbb{N} \to \mathbb{N}$, using Turing machines (and variants thereof) as the base computational model [1].

A notable exception is the study of type-2 functionals, *i.e.* functions of type $(\mathbb{N} \to \mathbb{N}) \to \mathbb{N} \to \mathbb{N}$, where many results are available and a clear definition of type-2 *polynomial time* computability is accepted by the community. This definition is *robust* as it can be either given using oracle Turing machines (OTM), or implicitly, *e.g.* via a function algebra in the style of Cobham. The resulting class is often called *basic feasible functionals* (BFF) [3].

In this talk, we will concentrate on the following natural question: how can we define type-2 *logarithmic space*? As in the first-order case, dealing with logarithmic space is harder than dealing with polynomial time. We concentrate on machine models, leaving the definition of an implicit characterization (which is arguably simpler) for future work. When dealing with logarithmic space, computational resources are very constrained and the design of the model is crucial. Indeed, the idea is that a LOGSPACE (or L) algorithm can only store a *constant* number of *pointers* to its immutable input. Designing type-2 LOGSPACE means understanding how to constrain oracle Turing machines to work in LOGSPACE, in a reasonable way. Indeed, we would like to satisfy some desiderata, in particular the closure of this class w.r.t. composition. Since we are in the type-2 world, this means verifying both type-0 and type-1 composition, as follows:

Definition 1 (Composition)**.** Let $\mathcal{C}$ be a class of type-2 functionals.

- $\mathcal{C}$ is closed under type-0 composition if $F \in \mathcal{C}$ and $G \in \mathcal{C}$ imply $H(f, x) := F(f, G(f, x)) \in \mathcal{C}$.
- $\mathcal{C}$ is closed under type-1 composition if $F \in \mathcal{C}$ and $G \in \mathcal{C}$ imply $H(f, x) := F(\lambda y.G(f, y), x) \in \mathcal{C}$.

Why is this hard? Basically, type-0 composition is standard (first-order) function composition, while type-1 composition is closure by oracles. Then, let us consider the class FLOGSPACE (or FL) of the functions computable in logarithmic space. While FL is indeed type-0 closed, we have that $\mathsf{FL}^{\mathsf{FL}} = \mathsf{FP}$ [5]. This means that adding the possibility to query an FL oracle dramatically increases the computational power, thus invalidating the type-1 compositionality of FL.

The Machine Model. We start by defining our variant over oracle Turing machines. We will discuss the non-standard aspects below.

Definition 2. A tape stack oracle Turing machine (SOTM) M consists of:

- a two-way, read-only tape;
- a two-way, read/write working tape;
- a one-way, write-only output tape;
- a pushdown stack of pair of tapes $\langle Q_k, A_k \rangle$ such that for the active tapes $\langle Q_{\text{top}}, A_{\text{top}} \rangle$:
  - Q_{top} is one-way and write-only;
  - A_{top} is one-way and read-only;
  - each time M writes a symbol to Q_{top}, the content of A_{top} is erased;
- a finite state control including four special states `PUSH`, `PULL`, `QUERY`, `ANSWER`.

A few explanations are in order. The behavior of SOTM is the one of usual OTM, *i.e.* when the `QUERY` is reached the content x of the active query tape Q_{top} is read, the state changes to `ANSWER` and the string $f(x)$, where f is the oracle, is immediately written on the answer tape A_{top}. We also require that a sort of "independence" between answers and queries. Once we start to write a query, the answer tape is immediately wiped. The novelty here is the presence of a *stack* of query and answer tapes, inspired by Cook's AuxPDA [2] and Wilson's *oracle stacks* [6]. This stack is managed through the states `PUSH` and `PULL` that create/destroy new pairs of tapes, using a stack-based discipline. This way, only the top most pair can be written/read.

Easy Feasible Functionals. Before defining our new complexity class we need a technical definition that constraints the power of oracles.

Definition 3 (Polynomially-Bounded Oracle)**.** A type-1 functional oracle f is polynomially-bounded if there exists a polynomial P such that for all queries y we have $|f(y)| \leq P(|y|)$.

We are now able to define the class which is the type-2 equivalent of FLOGSPACE.

Definition 4 (Easy Feasible Functionals (EFF))**.** A functional F is in EFF if and only if there exists a SOTM M that computes F such that for each type-0 input x and type-1 polynomially-bounded oracle f[1], the maximum number of cells written on the working tape is $\mathcal{O}(\log |x|)$ and the maximum height of the tape stack is $\mathcal{O}(1)$.

The interesting thing to note here is that, unlike the case of BFF, we do not need second-order polynomials. This is for a very precise reason: while polynomials perfectly compose, this is not the case for logarithms of polynomials. Moreover, please note the restriction to the maximum stack height to a constant.

Theorem 1.

1. EFF *are closed under type-0 composition.*
2. EFF *are closed under type-1 composition.*

Proof ingredients.

1. Here the constant height tape stack plays a crucial role in the simulation.
2. Here we exploit (i) the fact that answer tapes get erased when we start to write a new query and (ii) the polynomial bound on oracles. □

[1]The restriction to polynomially-bounded oracles could seem either very strong or arbitrary. Actually, it this restriction turns out to be equivalent to devising a SOTM that can ask for the content of the oracle query $f(y)$ bit-by-bit, via pointers. This mechanism is standard when dealing with LOGSPACE composition.

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Uniform incompleteness in proof-theoretic semantics: consistent bases and weakly classical meta-logic

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Proof-theoretic semantics (PTS) is an approach to meaning and to (logical) validity put forward by Prawitz in the 1970s [15, 16].

The core-notion of PTS is, not *truth*, but *proof*. So, the "models" of PTS are proof-systems—called *atomic bases*—consisting of atomic *higher-level rules* in the style of Schroeder-Heister [18, 19], namely, rules of a given level k, where premises and conclusions are atoms, and where atomic rules of level $k-2$ can be discharged.

In Prawitz's original writings, the notion of validity is primarily referred to *arguments*, that is, tree-formed structures whose nodes are labelled by formulas, and whose edges stand for *arbitrary* inferences (some of which may operate dischargements). Prawitz endorses Gentzen's tenet that introduction rules of Natural Deduction (ND) define the meaning of the logical constants, whereas elimination rules can be justified based on this meaning determination [4]. The exploitation generalises Prawitz's own normalisation theory for ND [13, 15], which shows that "local peaks" in derivations—i.e., formulas that occur in the derivations at the same time as conclusions of introductions and as major premises of eliminations—can be removed by applying suitable *reductions*, so as to turn the derivations into normal forms, namely, derivations with no "local peaks".

Sticking to the idea that introductions are meaning-constitutive, thus valid by default, an argument $\mathscr{D}$ which is *closed*—i.e., depends on no undischarged assumptions—is *valid* on an atomic base $\mathfrak{B}$ when it can be turned, modulo a set of reductions $\mathfrak{J}$, into a derivation from $\mathfrak{B}$, if its conclusion is atomic or, if its conclusion is compound, into an argument $\mathscr{D}'$ which is *canonical*—i.e., ends by introduction—with immediate sub-arguments valid on $\mathfrak{B}$ modulo $\mathfrak{J}$. When $\mathscr{D}$ is *open*—i.e., depends on undischarged assumptions—its validity is defined *monotonically* or *non-monotonically*. In the first case one requires the validity on any $\mathfrak{B}' \supseteq \mathfrak{B}$ modulo any $\mathfrak{J}' \supseteq \mathfrak{J}$ of any closure of $\mathscr{D}$ obtained by replacing the undischarged assumptions by closed arguments for them that are valid on $\mathfrak{B}'$ modulo $\mathfrak{J}'$. In the second case, one simply drops out quantification over extensions of $\mathfrak{B}$. These "purely" Prawitzian versions of PTS are called *monotonic* and *non-monotonic proof-theoretic validity* (mP-tV and nP-tV respectively).

A crucial role in both mP-tV and nP-tV is played by reductions. The latter are understood by Prawitz as constructive functions Φ from non-canonical arguments onto arguments, such that, when Φ is defined on $\mathscr{D}$: (a) if $\mathscr{D}$ is from assumptions Γ to conclusion A, then $\Phi(\mathscr{D})$ is from $\Gamma' \subseteq \Gamma$ to A; (b) Φ is defined on any argument obtained from $\mathscr{D}$ by replacing undischarged assumptions by arguments for them, and the result of an application of Φ to such an argument is the same as that of doing the same replacement on $\Phi(\mathscr{D})$. Importantly, it is part of the reductions being constructive that they are *base-independent*, i.e., that they provide rules for rewriting arguments which do not depend on the behaviour of specific inputs on specific atomic bases.

The Prawitzian PTS which most scholars work with today, due do Sandqvist [17], and de Campos Sanz, Piecha and Schroeder-Heister [3, 11, 12, 20], is however different from mP-tV and nP-tV. These approaches do not appeal to arguments and reductions any longer, but to a notion of consequence over an atomic base defined by direct induction on the complexities of Γ and A. Also in this case, consequence of A from $\Gamma \neq \emptyset$ can be defined monotonically or non-monotonically, so these variants are called *monotonic* or *non-monotonic base-extension semantics* (mB-eS and nB-eS respectively).

In 1973, Prawitz put forward a conjecture of completeness of intuitionistic logic over nB-eS, nowadays known simply as *Prawitz's conjecture*. The conjecture can be of course formulated also for mP-tV, mB-eS and nB-eS, and has been fully refuted for the last two frameworks by de Campos Sanz, Piecha and Schroeder-Heister [3, 11, 12, 21]. But de Campos Sanz, Piecha and Schroeder-Heister's proofs do not carry over to mP-tV and nP-tV, as an embedding of mB-eS and nB-eS onto them forces reductions to be base-dependent—when base-independence is dropped out, the embedding does go through, and the incompleteness proofs do apply [5, 6, 8, 7, 9, 10].

A proof of intuitionistic incompleteness over mP-tV, so with base-independent reductions, has been later on provided by Piccolomini d'Aragona [9]—via a similar result for a notion of validity based on BHK-constructions by de Campos Sanz and Piecha [2]. The same was achieved recently by Piccolomini d'Aragona and Prawitz for nB-eS too [10]. Hence, Prawitz's conjecture can be said to have been refuted everywhere. However, in my talk I show that more can be done, in particular, that interesting and fruitful variants of the incompleteness theorems of both [9] and [10] can be provided.

Piccolomini d'Aragona's incompleteness proof for mP-tV has a shortcoming: it applies only when atomic bases, which are allowed to be inconsistent, contain all the instances of *ex falso* from the atomic constant symbol for absurdity $\bot$ to any atomic A, or when $\bot$ is a 0-ary connective with the clause that it is valid on an atomic base if and only if all atoms are valid on that very base. Barroso Nascimento, Pereira and Pimentel have recently drawn attention to the relevance and depth of a variant of PTS where atomic bases (and their extensions) are required to be always consistent [1]—a variant which, incidentally, is historically very accurate, since this is how Prawitz has conceived of PTS since 1970 [14]. A first result of my talk is thus that, by suitably modifying the incompleteness proof of [9], intuitionistic logic can be shown to be incomplete also over mP-tV with consistent atomic bases.

Similarly to the incompleteness proof in [10], this first result uses excluded middle in the meta-logic. In particular, it reads as follows.

Theorem 1. *With A, B and C atomic, and with classical meta-logic, the rule*

$$\frac{A \to B \vee C}{(A \to B) \vee (A \to C)}$$

is logically valid in mP-tV when atomic bases are assumed to be always consistent.

As observed by Piecha and Schroeder-Heister in [12, 21], this is enough for refuting Prawitz's conjecture, since a classical proof of incompleteness rules out a constructive proof of completeness. But this does not mean that a constructive proof of incompleteness would be of no value—on the contrary, it would be a very valuable achievement. This remark is important for the second result of my talk, which pertains instead to Piccolomini d'Aragona and Prawitz's incompleteness theorem for nB-eS.

While the proof of this theorem has no shortcomings, it uses, as said, excluded middle in the meta-logic. More precisely, it assumes that for every $\mathfrak{B}$ and every atom A, either A is provable in $\mathfrak{B}$ from no undischarged assumptions or it is not. As a step towards finding a constructive proof of incompleteness for nB-eS, I shall prove that intuitionistic logic is incomplete over nB-eS also with a weaker (albeit still non-constructive) meta-logic, i.e., with *weak* excluded middle in the following form: for every $\mathfrak{B}$ and every atom A, either A is not provable from undischarged assumptions in $\mathfrak{B}$, or it is not. In particular, we have what follows.

Theorem 2. *With A, B and C atomic, and with weak excluded middle in the meta-logic, the rule*

$$\frac{\neg\neg A \to B \vee C}{(\neg\neg A \to B) \vee (\neg\neg A \to C)}$$

is logically valid in nB-eS when atomic bases are assumed to be always consistent.

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Predicates and Comprehension schema in Relational Doctrines

Algebraic logic is often concerned with the study of lattices that are understood as models of certain logics. Among the most well studied classes of lattices there are Boolean and Heyting algebras, that model, respectively, the classical and the intuitionistic calculus of propositions. One can restrict the focus to other lattices, but, in general, it is convenient to work at least with meet-semilattices. Indeed, if one wants to have Lindenbaum-Tarski algebras (of the fragment of propositional logic with which one is working) a natural way to encode a sequent as an inequality between two elements of a lattice is to take the meet of the premises. In this respect, and unless one is interesed in substructural logics, meet-semilattices have the minimal structure to be considered as lattices of propositions.

Soon after the introduction of Boolean algebras, the work of De Morgan, of Pierce and of Schroder brought to the introduction of relation algebras. If Boolean algebras are meant to model propositions, also called unary relations, relation algebras are meant to model binary relations (the archetypal example of Boolean algebras over a set U is $\mathcal{P}(U)$, while that of relation algebras is $\mathcal{P}(U \times U)$). Relation algebras are Boolean algebras extended by operations and constants that are specific to the calculus of binary relations. These are the composition of relations, the diagonal relation and the converse of a relation and they make the undrelying Boolean algebra an involutive ordered monoid. As before, also for relation algebras, one can restrict the focus on different lattices provided that they have the minimal structure to be considered lattices of relations, i.e. provided that they are involutive ordered monoids.

A natural question is how to recover unary relations in relational settings, i.e. how one can select, in an involutive ordered monoid, elements that can be understood as unary relations.

A standard approach is to take the co-reflexive relations, that is, relations included in the diagonal. This choice behaves very well in relation algebras as it meets some desiderata that one expects to hold for those relations that are actually unary. For example, unary relations should not be affected by the operation of taking the converse, i.e. a unary relation should be symmetric. Another property concerns the composition: since the composition of relations can be understood as the relative analogue of conjunction (also in virtue of its distributivity over finite joins) [**?**], the composition of two binary relations that are unary should collapse to their meet (then the diagonal collapses to the top element). This happens if and only if unary relations are also co-transitive, that is if a relation r is contained in the composition of r with itself[†]. As mentioned before, in relation algebras co-reflexive relations can be proved to be also symmetric and co-transitive, hence *co-equivalence relations*, but the proof crucially relies on a property (namely, Tarski's equational reformulation of DeMorgan's laws [**?**]) which is not available in more general relational contexts, such as in generic involutive ordered monoids [‡].

In this talk, we propose to define unary relations directly as co-equivalence relations. We discuss the benefits of our definition in the more general setting of doctrines.

Doctrines were introduced in the late sixties by F. W. Lawvere. They are families of lattices, indexed by certain categories, that give categorical models of (fragments of) the first order predicate calculus. More specifically a doctrine takes the form of a functor

$$\mathbf{C}^{\mathrm{op}} \xrightarrow{P} \mathbf{Pos}$$

where an object X of $\mathbf{C}$ is understood as if it 'contains' some variables and $P(X)$ is understood as a lattice of formulas whose variables are in X. The domain category of P is $\mathbf{C}^{\mathrm{op}}$, that is P is contravariant. This captures the contravariant nature of substitution: indeed if $f\colon X \to Y$ is understood as an assignment that maps variables in X to variables in Y, then every formula on Y can be turned, by substitution, into a formula on X.

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*Joint work with F. Dagnino (University of Genova)

†Co-transitive relations are also called *interpolative* or *dense*. Indeed r is co-trasitive if xry if and and only if there is z such that xrz and zry.

‡A first consequence of generalty is that co-equivalence relations need not be closed under composition.

As for lattices, there are several kinds of doctrines depending on the fragment of predicate logic that one wants to model.

On the relational side, doctrines modelling the calculus of relations are introduced in [**?**] and called relational doctrines. They are functors of the form

$$(\mathbf{C} \times \mathbf{C})^{\mathrm{op}} \xrightarrow{R} \mathbf{Pos}$$

where each $R(A, B)$ is understood as the lattice of relations with domain A and codomain B. More precisely, the posets in the image of R are endowed with the binary operation of composition of relations that sends $\alpha \in R(A, B)$ and $\beta \in R(B, C)$ to $\alpha\,;\beta \in R(A, C)$, with a unary operation of taking the converse sending $\alpha \in R(A, B)$ to $\alpha^{\perp} \in R(B, A)$ and a constant $\mathsf{d}_A \in R(A, A)$ representing the diagonal relation over A. These operations satisfy the following equations

$$\begin{array}{lll} \alpha\,;(\beta\,;\gamma) = (\alpha\,;\beta)\,;\gamma & \mathsf{d}_A\,;\alpha = \alpha & \alpha\,;\mathsf{d}_B = \alpha \\ (\alpha\,;\beta)^{\perp} = \beta^{\perp}\,;\alpha^{\perp} & \mathsf{d}_A^{\perp} = \mathsf{d}_A & \alpha^{\perp\perp} = \alpha \end{array}$$

Relational doctrines generalises involutive ordered monoids (hence relation algebras), heterogeneous relation algebras in the sense of [**?**] and those Lawvere's doctrines that model at least the $(\exists, =, \wedge, \top)$-fragment of predicate logic, called elementary existential in [**?**]. Moreover they allow also to consider examples based on metric spaces in such a way that the diagonal relation over a metric space (A, d_A) is the distance d_A (providing in this way a logical framework for the so-called quantitative reasoning [**?**], i.e. a logic where equality predicates can be interpreted as distances).

For a given relational doctrine R over a category $\mathbf{C}$, we define unary relations, also called *predicates*, as co-equivalence relations: i.e. α is a predicate over X in $\mathbf{C}$ if $\alpha \in R(X, X)$ satisfies

$$\text{co-reflexivity: } \alpha \leq \mathsf{d}_A \quad \text{co-symmetry: } \alpha^{\perp} \leq \alpha \quad \text{co-transitivity: } \alpha \leq \alpha\,;\alpha$$

A notion which is typically related to predicates is that of comprehension. The notion of comprehension was formalised in the context of doctrines by Lawvere in [**?**]. In his approach, Lawvere made an extensive use of the language of adjoints to capture the structural behaviour of the comprehension schema. Maietti and Rosolini weaken Lawvere's original formulation leading to a generalisation that allows to consider a class of doctrines that is wider than the one considered by Lawvere. Here we bring the generalisation a step further defining comprehension for co-equivalence relations for a given relational doctrine.

Since the definition of predicates as co-equivalence relations means that predicates and equivalence relations are dual concepts, one is led to ask if this duality can be pushed a bit further. For example, one can wonder if there is a connection between relational doctrines with comprehension and those with quotients (that is the relational doctrines in which it is possible to compute quotients for internal equivalence relations).

We give a positive answer, showing that every relational doctrine R determines a dual doctrine R^{op}, such that $(R^{\mathrm{op}})^{\mathrm{op}}$ is again R and such that a relation is a predicate in R if and only if it is an equivalence relation in R^{op}. The two main results that we will present in this seminar are stated in the following propositions.

Proposition. A relational doctrine R has comprehension if and only if R^{op} has quotients.

Proposition. The category of relational doctrines with comprehension is equivalent to the category of relational doctrines with quotients.

The second proposition gives to the connection between comprehensions and quotients the flavour of a duality as the equivalence is obtained by mapping a relational doctrine R to its dual R^{op}. The advantage of such a duality becomes clear if one considers that quotients have been largely studied in mathematics and the literature on quotients and their properties is quite wide, even if one restricts the focus on the part of the literature that concerns doctrines.Thus, one can derive many properties on doctrines with comprehension from the corresponding (dual) ones on doctrines with quotients.

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Generating Functions for Probabilistic Programs via the Weighted Relational Model of Linear Logic

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1 Introduction

The recent years have seen a surge of interest towards probabilistic programming. Probabilistic programs pose difficult challenges already when one only considers discrete probabilistic choices (i.e. Bernoulli random variables): as opposed to deterministic programs, each execution of a probabilistic program can give a different result both in term of convergence and of its output; hence, while for non-deterministic programs one can study *qualitative* properties (does the program terminate or not?), for probabilistic programs one has to consider *quantitative* generalizations of such properties (what is the probability that the program terminates?).

However, quantitative properties can be difficult or even intractable in many contexts. For instance, the literature on *probabilistic model-checking* has studied the problems of determining the probability of termination or the expected running time for different computational models, leading to several results, both positive and negative, regarding the decidability and tractability of the underlying verification problem (see [1] for a recent survey). Beyond termination, one can consider the problem of *exact inference*, that is, of finding the posterior distribution of the values computed by a program. While efficient methods are known for finitary programs, in the presence of *unbounded* iteration (and hence the possibility of non-termination), inference is undecidable [19], and a clear picture of the decidable or tractable cases is still missing.

Properties like the probability of termination or the expected running time are often sensitive to how much the program is able to *duplicate* its inputs. This is why ideas and methods from linear logic have been widely applied in this area. In particular, the recent literature has explored methods based on the weighted relational model [12] and the related probabilistic coherent spaces (PCOH) [4]. Our starting point is indeed the basic observation that the interpretations of programs in both such models can be represented as formal power series [2], [13]. For example, the interpretation of a program $\mathcal{G}$ that can reduce to a finite set of outputs $[\![A]\!] = \{a_1, \dots a_k\}$ yields a family of n *generating functions* $A_k(z) = \sum_n \mathbf{P}(\mathcal{G} \to^n a_k) z^n$, where $\mathbf{P}(\mathcal{G} \to^n a_k)$ indicates the probability of $\mathcal{G}$ reducing to the value a_k in n steps.

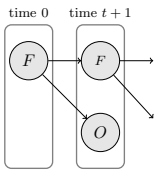
Our general goal is to explore the use of the concept of generating function, that is classical in enumerative and analytic combinatorics [5, 6], to identify classes of probabilistic programs whose quantitative properties can be extracted, in a computable way, from their interpretation in the weighted relational model or PCOH. Notably, while the combinatorial approach has already been explored in the field of imperative programming [7], through these linear logic models we manage to extend it to *higher-order* programming languages.

In the recent paper [11] we provided a first demonstration of this method by reproving (and extending) a result of [15] about probabilistic model checking using algebraic power series. In this abstract we first briefly recall this result, and then sketch a few new results and perspectives arising from this method.

$$S = Fe,$$
$$Fx = x \oplus_{\frac{1}{2}} F(Fx).$$

(a) PHORS computing a simple random walk: starting from Fe the number of F may either increase or decrease by one at each reduction, and termination occurs when this number reaches zero.

$$S = HFO$$
$$F = \mathsf{True} \oplus_q \mathsf{False}$$
$$O = \mathsf{True} \oplus_p \mathsf{False}$$
$$Hxy = \text{if } x \text{ then } H_{\mathrm{True}}xy \text{ else } H_{\mathrm{False}}xy$$
$$H_{\mathrm{True}}xy = Hxy \oplus_r O$$
$$H_{\mathrm{False}}xy = Hxy \oplus_s O$$



(b) A Boolean PHORS and the corresponding iterated model: the functions $H, H_{\mathsf{True}}, H_{\mathsf{False}}$ describe the propagation of the model which, at each iteration, queries the Boolean variable F and, depending on the result, either produces an observable O or queries F again.

Figure 1: Examples of PHORS.

2 PHORS, aka. the Probabilistic λY-Calculus

In *higher-order* model-checking one studies the decidability and tractability of verifying properties of programs described in languages arising from the λ-calculus. As the termination property is undecidable for this language, the literature has mostly focused on restricted languages like the HORS (Higher Order Recursion Schemes), due to a seminal result [17] stating that all properties of HORS expressible in monadic second order logic (including termination) are decidable. A HORS is defined as the fixpoint of a finite system of recursive equations of the form $Fx_1 \dots x_n = t$. The functional variables F defined by such equations are called the *non-terminal symbols* of the PHORS. In the *probabilistic* HORS, or PHORS [8], equations $Fx_1 \dots x_n = t_L \oplus_p t_R$ involve probabilistic choices $\oplus_p$. The execution of a (P)HORS starts from a *source* non-terminal S and goes on by unrolling the equations. For example, the PHORS in Fig. 1a simulates a simple random walk. The (P)HORS are equivalent to the simply typed λY-calculus [18], that is, the simply typed λ-calculus enriched with fixpoints $Y : (\sigma \to \sigma) \to \sigma$ at all types σ (and with the choice operators $\oplus_p$ in the probabilistic case).

While qualitative properties like termination are decidable for the HORS, the corresponding quantitative properties of PHORS are not: establishing if a PHORS terminates with maximal probability (*almost sure termination*, AST) is undecidable [8]. It is then interesting to study for which classes of PHORS problems like AST (or the related PAST problem, asking whether the expected running time is finite) are tractable or, at least, decidable.

To study exact inference we also consider an extension of the PHORS with a type of Booleans and if-then-else, equivalent to a probabilistic version of finite PCF [16]. This language represents both finite graphical models (e.g. Bayesian networks) and *unbounded* models [9, Ch. 6], iterating some basic network template, as in the example in Fig. 1b. A Boolean PHORS $\mathcal{G}$ reduces with probability p to False and with probability q to True (with $p+q \leq 1$), noted $\mathcal{G} \sim \mathrm{Bernoulli}(p, q)$. The (generally undecidable) problem of exact inference is to find these p, q.

3 Finiteness and Algebraicity

The weighted relational model of linear logic is a well-studied denotational semantics in which a program $M : \sigma \to \tau$ is interpreted as a matrix $[\![M]\!] : !|\![\sigma]\!| \times |\![\tau]\!| \to \mathcal{Q}$ with coefficients in

a continuous semiring $\mathcal{Q}$, where $[\![\sigma]\!], [\![\tau]\!]$ are countable sets and $!X$ denotes the set of finite multisets on X. Our starting observation is that this matrix can be equivalently described as a family of formal power series $[\![M]\!](x)_b = \sum_{\mu \in ![\![\sigma]\!]} [\![M]\!]_{\mu,b} x^\mu$ (for $b \in [\![\tau]\!]$), where for all $x \in \mathcal{Q}^{[\![\sigma]\!]}$, $x^\mu = \prod_{a \in [\![\sigma]\!]} x_a^{\mu(a)}$.

Interpreting a PHORS in this model corresponds then to finding the *least fixpoint* of an *infinite* system $[\![\mathcal{G}]\!] = \Phi([\![\mathcal{G}]\!])$ of equations between power series. In particular, by choosing $\mathcal{Q} = \mathbb{R}_{\geq 0}^{+\infty}\{\!\{z\}\!\}$, the semiring of formal power series in z with positive real coefficients, the interpretation of the source symbol S of a PHORS yields a generating function $[\![S]\!](z) = \sum_{n \geq 0} \mathbf{P}(\mathcal{G} \Downarrow_n) z^n$ describing the probability of termination in n reduction steps.

Unfortunately, the infinitary systems produced by the PHORS are not, in general, tractable with algebraic or computational tools: this model is not, globally, computable. However, what if we could ensure, by some appropriate restrictions on the PHORS $\mathcal{G}$, that the resulting equational system is equivalent to some *finite* polynomial system? This is the approach we follow in [11], where we introduce a semantic condition of *finiteness* to this effect. For the technical definition of a *finitary family* see [11]; we here report only its fundamental property:

Proposition 1. *Suppose $(\Phi_i(\vec{x}, \vec{y}))_{i \in I}$ is a finitary family of power series. If $r_i(\vec{y})$ is the least solution of the system $r_i(\vec{y}) = \Phi_i(r_i(\vec{y}), \vec{y})$, then* all but finitely many *of the $r_i(\vec{y})$ are equal to 0 and the remaining ones are solution of a finite family of polynomial equations $(r_{j_\ell}(\vec{x}) = p_\ell(r_{j_1}(\vec{x}), \ldots r_{j_n}(\vec{x}), \vec{y}))_{\ell \in \{1 \ldots n\}}$.*

Now, the main point is to identify a class of PHORS whose interpretation satisfies the previous condition. We do this by introducing the class of *bounded PHORS* (PBHORS$^\infty$), which extends the *affine PHORS* from [15]. The bounded PHORS are defined using a linear type system involving *graded exponentials* (where a graded type of the form $!_k\varphi \multimap \psi$ describes a program that can use an input of type φ *at most k times* to produce an output of type ψ). More precisely, a type judgement in this system is of the form $\mathcal{N} \mid \Delta \vdash t : \sigma$, where:

- $\mathcal{N}$ is a *non-linear* environment for the non-terminal symbols;
- Δ is a *graded* environment for the other variables, comprising *both* finite and infinite grades.

The type system is designed to do two things: first, restrict the use of variables with an infinite grade (those which can be duplicated at will), as these may only play the role of the *parameters* $\vec{y}$ in the power series $r_i(\vec{y})$ above; second, enforce a *fixpoint condition* ensuring that for any equation $Fx_1 \ldots x_n = t$, the non-terminal function symbol F and the term $\lambda x_1 \ldots x_n.t$ agree on the grades assigned to $x_1, \ldots, x_n$; this condition warrants that the least fixpoint $r_i(\vec{y})$ is determined by only a *finite* number of conditions. This leads to the following:

Theorem 1. *For all PBHORS$^\infty$ $\mathcal{G}$, the interpretation $[\![\mathcal{G}]\!]$ is the least solution of a finite polynomial system. In particular, the generating function $[\![S]\!](z) = \sum_{i \geq 0} \mathbf{P}(\mathcal{G} \Downarrow_n) z^n$ is algebraic (i.e. there is a polynomial $p(z, w)$ with rational coefficients such that $p(z, [\![S]\!](z)) = 0$).*

Using Tarski's decidability of the first-order existential theory of the reals, we obtain then:

Theorem 2. *For all PBHORS$^\infty$, the AST and PAST problems are decidable (EXPTIME if order is fixed).*

Example 1. *The PHORS in Fig. 1a is a PBHORS$^\infty$, its generating function $[\![S]\!](z)$ annihilates the polynomial $p(z, y) = \frac{1}{2}y^2 z - y + \frac{1}{2}z$, which can be used to show that it is AST but not PAST.*

4 New Results and Perspectives

We now list a few new applications and perspectives of the results in [11].

Rational PHORS It is well-known that algebraic numbers are computable, in the sense of being approximable to arbitrary degree. Similarly, algebraic power series come with several well-known exact or approximate computational methods. However, just like rational numbers are computable in an exact way, stronger (i.e. more tractable) methods exist when considering rational power series, that is power series that can be expressed as a ratio of two polynomials.

Starting from the type system described above, it is not difficult to design a more restricted discipline which enforces rationality. Indeed, it is well-known that rational power series $r(\vec{y}) = \frac{p(\vec{y})}{q(\vec{y})}$ can be characterized as the solution of *linear* recursive systems $X_i = a_i(\vec{y})X_i + b_i(\vec{y})$ [10]. Such systems can be solved effectively (and efficiently) by means of the Gauss algorithm [14].

In our system, this linearity constraint can be ensured by restricting the discipline of non-terminal symbols: in the typing context $\mathcal{N} \mid \Delta \vdash t : \sigma$ we now require that the variables in $\mathcal{N}$ be used in an *affine* way, that is, possibly discarded but never duplicated. Call PRatHORS$^\infty$ the class of PHORS typable in this restricted system. We then have the following:

Theorem 3. *For all PRatHORS$^\infty$ $\mathcal{G}$, the interpretation $[\![\mathcal{G}]\!]$ is the least solution of a finite linear system. In particular, the generating function $[\![S]\!](z) = \sum_{i \geq 0} \mathbf{P}(\mathcal{G} \Downarrow_n) z^n$ is rational and its closed form can be computed in polynomial time (when order is fixed). Hence, the probability of termination is a rational number which can be computed explicitly from the typing derivations.*

Exact Inference As suggested above, given the computational nature of rational and algebraic power series, we can exploit Theorems 1 and 3 to devise solutions to the exact inference problem for a Boolean PHORS.

Theorem 4. *For all Boolean PHORS $\mathcal{G} \sim$ Bernoulli(p, q):*

- *if $\mathcal{G}$ is a PBHORS$^\infty$, then for any given $0 < \epsilon \in \mathbb{Q}$ we can compute approximations $p', q' \in \mathbb{Q}$ such that $|p - p'| < \epsilon, |q - q'| < \epsilon$;*
- *if $\mathcal{G}$ is a PRatHORS$^\infty$, then p and q are rational and can be computed effectively.*

Example 2. *The PHORS in Fig. 1b is a PRatHORS$^\infty$, with rational generating functions $[\![S]\!]_{\mathrm{True}}(z) = p\frac{\alpha z^3}{1-\beta z^2}$ and $[\![S]\!]_{\mathrm{False}}(z) = \overline{p}\frac{\alpha z^3}{1-\beta z^2}$, where $\alpha = q\overline{r} + \overline{q}\overline{s}$ and $\beta = qr + \overline{q}s$, yielding, for e.g. $p = q = \frac{1}{3}$ and $r = s = \frac{1}{2}$, $\mathcal{G} \sim$ Bernoulli$(\frac{1}{3}, \frac{2}{3})$.*

Asymptotic Estimates The algebraicity of the generating function of a PHORS is not only useful to answer exact problems like AST, but also to estimate *how fast* the program converges. We provide here a sketch of how, using the methods from analytic combinatorics [5], it might be possible to make such estimates. If the radius of convergence of the power series $[\![\mathcal{G}]\!](z)$ is $\rho > 1$, then $\mathbf{P}(\mathcal{G} \Downarrow_n) = O(1/\rho^n)$. On the other side, if $\rho = 1$, (the case $\rho < 1$ can be excluded by standard convergence results), the asymptotic behaviors is regulated by the nature of the dominant (i.e. smallest in modulus) singularities of $[\![\mathcal{G}]\!](z)$: for example, if $[\![S]\!](z)$ has a unique dominant singularity of 'square root type' (i.e. it has a Puiseaux expansion of the form $[\![S]\!](z) = \sum_{k \geq 1} c_k(z-1)^{k/2}$), one obtains that $\mathbf{P}(\mathcal{G} \Downarrow_n) = O(n^{3/2})$.

While the problem of finding the radius of convergence and the Puiseaux expansions is generally not computable, when the function is algebraic (so there exists an irreducible polynomial $p(z, w)$ s.t $p(z, [\![S]\!](z)) = 0$) singularities are located at the points (z, w) for which $\frac{\partial p(z,w)}{\partial w} = 0$ and algorithms exist to compute both [5], [3]. We obtain for example that:

Proposition 2. *Let $\mathcal{G}$ be a PBHORS$^\infty$ and let $p(z, w)$ be the irreducible annihilating polynomial of $[\![\mathcal{G}]\!](z)$ (that can be computed effectively). Let $\mathbf{P}(\mathcal{G} \Downarrow_{\leq n})$ be the probability of termination* in no more than n steps. *If for no $|z| < 1$ we can find a w s.t. $\frac{\partial p(z,w)}{\partial w} = 0$, then there exists a $r < 1$ s.t. $\mathbf{P}(\mathcal{G} \Downarrow) = \mathbf{P}(\mathcal{G} \Downarrow_{\leq n}) + O(r^{n+1})$ for $n \to +\infty$.*

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Coequivalence relations and descent in modal and superintuitionistic logic

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In algebraic logic it has long been a subject of interest to correspond categorical properties of classes of models, on one hand, to logical properties of calculi, especially definability properties. In [5, Chapter 4.9], Ghilardi and Zawadowski introduced the concept of a *coequivalence relation*, a formula $\rho(\overline{p}_1, \overline{p}_2)$ which, in some context $\tau(\overline{p})$, is proved to be an equivalence relation by a logic L. Coequivalence relations in a logic L correspond to r-equivalence relations in the category $\mathsf{Alg}(L)^{op}_{fp}$. Coequivalence formulas equivalent to formulas of the form,

$$\tau(\overline{p}_1) \wedge \tau(\overline{p}_2) \wedge \bigwedge_{i=1}^{m} \psi_i(\overline{p}_1) \leftrightarrow \psi_i(\overline{p}_2)$$

are said to *separate variables*. Separating coequivalence relations correspond to *effective* r-equivalence relations, i.e., those that arise from a quotient.

Definition 1. A logic L is said to have the *coequivalence separation property* (CoSP) if all of its coequivalence relations separate variables.

This property stands in analogy with Beth definability: whereas projective Beth says that a term proved to be functional can be made explicit, coequivalence separation says that equivalence classes in the logic can be picked out by formulas explicitly. Categorically, the conjunction of these two properties corresponds to the Barr-exactness of the category $\mathsf{Alg}(L)^{op}_{fp}$. Model theoretically, it is related to questions of uniform elimination of imaginaries [8], and it has also been considered in the context of database theory [2].

Ghilardi and Zawadowski [5, Chapter 4.8] showed that IPC fails to have the CoSP. Work of the second author [4] implies that several systems of modal logic, including the well-known system $\mathsf{S5}$, fail to have the CoSP. At the same time, Ghilardi and Zawadowski asked whether for systems such as IPC one has that $\mathsf{Alg}(L)^{op}_{fp}$ is *almost Barr-exact*.

Given $\mathbf{C}$ a category with pullbacks and $p : E \to B$ a morphism in $\mathbf{C}$, there is a pair of adjoint functors

$$p_! \dashv p^* : \mathbf{C}/B \longrightarrow \mathbf{C}/E$$

with p^* pulling back along p and with $p_!$ composing with p from the left. We denote the category of algebras for the monad induced by this adjunction by $\mathsf{Des}(p)$, the category of *descent data* for p. We denote by $\Phi^p \colon \mathbf{C}/B \to \mathsf{Des}(p)$ the comparison functor.

Definition 2. A morphism p is an *effective descent morphism* if Φ^p is an equivalence of categories.

In other words, effective descent morphisms allow for an algebraic description of the B-indexed families by means of the E-indexed families in $\mathbf{C}$. Effective descent morphisms are always regular epimorphisms. Moreover, if $\mathbf{C}$ is Barr-exact, the vice versa holds.

Definition 3. A regular category $\mathbf{C}$ is said to be *almost Barr-exact* if all regular epimorphisms are effective descent morphisms.

An equivalent condition for regular categories is that all equivalence relations arising out of descent data are effective. These properties have been studied in category theory [6], locale theory [7], and algebra [9]. We introduce a logical counterpart:

For ease of notation, given $\alpha(\overline{p})$, we write $\mathsf{Ext}(\alpha) = \{\beta(\overline{p}, \overline{q}) : \beta \vdash \alpha\}$.

Definition 4. Let $\gamma(\overline{p}, \overline{q}, \overline{r}) \in \mathsf{Ext}(\beta(\overline{p}, \overline{q}))$. A sequence $\overline{\xi}$ of formulas over the variables $\overline{p}, \overline{q}_0, \overline{q}_1, \overline{r}$, namely $\xi_r(\overline{p}, \overline{q}_0, \overline{q}_1, \overline{r})$ for each $r \in \overline{r}$, is called a sequence of *local transition terms* if the following hold:

1. (γ-compatibility): $\gamma(\overline{p}, \overline{q}_0, \overline{r}) \wedge \beta(\overline{p}, \overline{q}_1) \vdash_L \gamma(\overline{p}, \overline{q}_1, \overline{\xi})$.
2. (Identity): For each $r \in \overline{r}$, $\gamma(\overline{p}, \overline{q}, \overline{r}) \vdash_L r \leftrightarrow \xi_r(\overline{p}, \overline{q}, \overline{q}, \overline{r})$.
3. (Cocycle): For each $r \in \overline{r}$, $\gamma(\overline{p}, \overline{q}_0, \overline{r}) \wedge \beta(\overline{p}, \overline{q}_1) \wedge \beta(\overline{p}, \overline{q}_2) \vdash_L \xi_r(\overline{p}, \overline{q}_0, \overline{q}_2, \overline{r}) \leftrightarrow \xi_r(\overline{p}, \overline{q}_1, \overline{q}_2, \overline{\xi})$.

Definition 5. Let $\alpha(\overline{p})$, $\beta(\overline{p}, \overline{q}) \in \mathsf{Ext}(\alpha)$ and $\gamma(\overline{p}, \overline{q}, \overline{r}) \in \mathsf{Ext}(\beta)$, equipped with a sequence of local transition terms $\overline{\xi}$. The *local coequivalence relation* associated is the γ-coequivalence relation

$$\rho_{\gamma,\overline{\xi}} = \gamma(\overline{p}_1, \overline{q}_1, \overline{r}_1) \wedge \gamma(\overline{p}_2, \overline{q}_2, \overline{r}_2) \wedge \bigwedge_{p \in \overline{p}} (p_1 \leftrightarrow p_2) \wedge \bigwedge_{r \in \overline{r}} (\xi_r(\overline{p}_1, \overline{q}_1, \overline{q}_2, \overline{r}_1) \leftrightarrow r_2).$$

Definition 6. A logic L has the *local coequivalence separation property* (LCoSP) if for each $\alpha(\overline{p})$, $\beta(\overline{p}, \overline{q}) \in \mathsf{Ext}(\alpha)$, $\gamma(\overline{p}, \overline{q}, \overline{r}) \in \mathsf{Ext}(\beta)$ and a sequence of local transition terms $\overline{\xi}$, the local coequivalence relation $\rho_{\gamma,\overline{\xi}}$ separates variables.

Our basic bridge theorem is the following:

Theorem 7. *For L a logic with the Beth property, L has the local coequivalence separation property if and only if in $\mathsf{Alg}(L)^{op}_{fp}$ all descent equivalence relations are effective. If L has the right-uniform Maehara interpolation, then this is additionally equivalent to $\mathsf{Alg}(L)^{op}_{fp}$ being an almost Barr-exact category.*

Using this bridge theorem and that of Ghilardi and Zawadowski, we prove the following results concerning the CoSP and the LCoSP:

Theorem 8. *The following systems fail to have the* CoSP*: $\mathsf{S5}, \mathsf{IPC}, \mathsf{LC}$.*

Using a characterization of S5 free objects due to [1], and the Birkhoff adjunction, which represents them as $\mathsf{Fam}(\mathbf{FinSet}^{s}_{\neq\emptyset})$, families of non-empty finite sets with surjective maps, we prove the following:

Theorem 9. *The system $\mathsf{S5}$ has the* LCoSP*.*

In the setting of $\mathsf{S5}$ we also look at the *categorical Galois theory* [3] of $\mathsf{S5}$ frames. By exploiting the adjunction employed in our representation, we show:

Theorem 10. *In $\mathsf{S5}$ with respect to the Birkhoff adjunction, the morphisms of Galois descent are exactly the étale maps, i.e. the maps where the restriction to each cluster is an isomorphism.*

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Higher Polish groups at singular cardinals of countable cofinality

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A Polish group is a topological group whose topology is completely metrizable and separable. In classical descriptive set theory, there are several important results concerning this kind of spaces and their continuous actions on Polish spaces. One may ask how much important is the assumption 'separable' in this setting: understanding what happens if we replace that assumption with 'there exists a base of size at most λ' (where λ is a given uncountable cardinal) is the main goal of *generalized descriptive set theory*.

To this purpose, one may immediately note how the cofinality of λ plays a crucial role in this setting: for instance, if $\mathsf{cf}(\lambda) > \omega$, then the *higher Baire space* $\lambda^{\mathsf{cf}(\lambda)}$ is not even first countable (hence, neither metrizable) when equipped with the bounded topology. In particular, we have two different approaches:

- Develop a "cofinality agnostic" study, which relies on a certain generalization of the usual definition of metric that fits with the case of uncountable cofinalities;
- Restrict our study to the case of (strong limit) singular cardinals with countable cofinality (e.g. $\lambda = \beth_\omega$), in order to remain in the realm of metrizable spaces.

As spoiled in the title, our approach will be the second one, therefore the main objects of our interest are the λ-*Polish groups*, namely those topological groups whose topology is completely metrizable and admits a base of size at most λ, where λ is a singular cardinal and $\mathsf{cf}(\lambda) = \omega$.

One of the main issues with the singular case (and perhaps the main one) is that the higher Baire space λ^ω (as well as λ^λ, often our ambient space in our framework) is not even $\aleph_1$-Baire, in the sense that there are $\aleph_1$-many open dense sets whose intersection is empty. In particular, the "Baire category arguments" used in several proofs cannot be generalized by replacing every occurrence of ω with λ, and in fact it turns out that the analogous of several classic result is simply false in the general case. For instance, we lose the automatic continuity of homomorphisms: there are λ-Borel measurable homomorphisms between λ-Polish groups which are not continuous.

However, many other result can be proved to be true in the general case: in this talk, we introduce a subgroup of $Sym(\lambda)$, called the *bounded permutation group*,

$$S_b(\lambda) = \{f : \lambda \to \lambda \mid f \text{ is a bounded bijection}\},$$

where "bounded" means that whenever $A \subseteq \lambda$ is bounded, both the image $f[A]$ and the preimage $f^{-1}[A]$ remain bounded in λ, and we basically prove that

1. $S_b(\lambda)$ is the largest λ-Polish subgroup of $(Sym(\lambda), \mathsf{bd})$;
2. $S_b(\lambda)$ is universal for a suitable class of zero-dimensional λ-Polish groups, generalizing a well-known result of Becker and Kechris on $Sym(\omega)$;

Finally, we investigate "how far" is $S_b(\lambda)$ from $S(\lambda)$ by studying the intermediate closed subgroups between these two spaces. This is a joint work with Claudio Agostini.

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Ultraproducts of Galois rings

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A chain ring is a commutative unital ring whose ideals are linearly ordered by inclusion. Such rings naturally arise in several areas of algebra and number theory. Classical examples include valuation rings and their finite quotients. In particular, the ring of p-adic integers can be obtained as the inverse limit of a sequence of finite chain rings of characteristic p^k as $k \to \infty$. Finite chain rings also appear in coding theory, finite geometry, and the study of finite local algebraic structures.

In this work, we focus on a distinguished class of finite chain rings known as *Galois rings*, denoted by $GR(p^n, r)$, where p is a prime, $n \geq 1$ is the characteristic exponent, and r is the extension degree. The existence and structure of Galois rings were studied by Krull [3], Janusz [2], and Raghavendran [5]. These rings may be viewed both as unramified extensions of residue rings of the form $\mathbb{Z}/p^n\mathbb{Z}$ and as quotients of valuation rings of unramified extensions of p-adic fields. From an algebraic perspective, they constitute natural finite analogues of complete discrete valuation rings with finite residue field.

The main objective of this project is to investigate the asymptotic model-theoretic behavior of families of Galois rings obtained by allowing the parameters p, n, and r to vary. Our approach is based on the study of ultraproducts of such structures. Given a family of finite Galois rings and a non-principal ultrafilter, the corresponding ultraproduct provides an infinite structure encoding the asymptotic first-order theory of the family. More precisely, by Łoś' theorem, and since every non-principal ultrafilter contains the cofinite subsets of the index set, properties holding for all but finitely many rings of the family are reflected in the ultraproduct. In this sense, ultraproducts provide natural infinite models for asymptotic phenomena arising in finite algebraic structures.

We analyze these ultraproducts from the perspective of contemporary classification theory [4, 6], with particular attention to the interaction between algebraic properties of finite rings and model-theoretic tameness phenomena. Different asymptotic regimes appear to exhibit different model-theoretic behavior. For example, one may distinguish between bounded and unbounded characteristic exponent, fixed versus varying residue degree, or fixed versus varying residue characteristic. Understanding how these asymptotic parameters influence the resulting theories constitutes one of the main guiding questions of the project.

This perspective extends previous work of Bello Aguirre [1] on ultraproducts of residue rings $\mathbb{Z}/p^n\mathbb{Z}$. More broadly, the project fits into the general program of studying asymptotic algebraic phenomena through infinite model-theoretic constructions.

The results presented here are part of ongoing joint work with Paola D'Aquino (Università degli Studi della Campania "Luigi Vanvitelli").

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Relative Hyperdefinability of the Lie model Theorem

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Abstract

In [5, Theorem 4.2] Hrushovski proves the Lie model theorem in full generality with model theoretic methods. The theorem states that for every approximate group there exists a generalized definable locally compact model, which, simplifying, is a quasi-homomorphism from the group generated by the approximate subgroup to a locally compact group with some particular properties. In [6, Theorem 3.30] Pillay and Krupiński prove the same theorem using topological dynamics on a locally compact type space. In this paper we study the definability of the locally compact group image of the quasi-homomorphism in this second proof. We show that it is isomorphic as a topological group to a relatively hyperdefinable locally compact group.

Extended Abstract

Combinatorics extracts structure information from cardinalities. So an example of a combinatorial question in the group setting may be how far away a subset of a group is from being a group based on the cardinality of the set of products. One of the first notions of almost being a group is small doubling. We say that a set A of a group G with $|A| = n$ has small doubling if $|A^2| \leq K|A|$ for some K. Of course $|A^2| \geq n$ and $|A^2| = n$ if A is a subgroup. Moreover $|A^2| \leq \frac{n(n+1)}{2}$. It is not very easy to find examples with small K, but if we work in $\mathbb{Z}$ then the following holds:

Proposition 0.1. *Let $A \subseteq \mathbb{Z}$ be such that $|A| = n$ be an arithmetic progression. Then $|A^2| = 2n - 1 \leq 2n$.*

In the 1960s Freiman described all the subgroups of small doubling in $\mathbb{Z}$ in [2] connecting them with d-dimensional arithmetic progressions.

Definition 0.2 (d-dimensional arithmetic progression)**.** We say that P is a d-dimensional arithmetic progression if there are $x_1, \ldots, x_d$ elements of $\mathbb{Z}$ and $m_1, \ldots, m_d$ positive integers and

$$P = \left\{ x_0 + \sum_{i=0}^{d} \lambda_j x_j \;\middle|\; 0 \leq \lambda_j \leq m_j \right\}$$

If all the sums in P are distinct, we say that P is proper.

Theorem 0.3 (Freiman)**.** *Let $A \subseteq \mathbb{Z}$ be such that $|A| = n$ such that $|A^2| \leq Cn$. Then A is contained in a proper d-dimensional arithmetic progression of cardinality at most Kn where K and d depend only on C.*

Later, Green and Rusza extended the result for arbitrary abelian groups in [3] in 2007. The problem is that this definition of small doubling does not work as well for non abelian groups. This is the reason why Tao introduced approximate subgroups in [7].

Definition 0.4 (Symmetry)**.** Let G be a group. A subset $X \subseteq G$ is symmetric if it contains the identity and $X = X^{-1}$.

Definition 0.5 (k-approximate group)**.** Let G be a group. A k-approximate group for some cardinal k is a subset $X \subseteq G$ which is symmetric and such that $X^2 = \{xy \mid x, y \in X\} \subseteq EX$ where $E \subseteq G$ is a set of cardinality k.

Most of the work in this area focused on classifying approximate subgroups up to commensurability.

Definition 0.6 (Commensurability)**.** Two subsets X, Y of a group G are commensurable if each is covered by finitely many translates of the other.

The complete classification of finite approximate subgroup was reached in 2012 by Breuillard, Green, and Tao in [1]. It relies on the Lie model theorem, proved by Hrushovski with model theoretic methods in [4].
The interest in approximate subgroups by model theorists comes from the fact that the notion of k-approximate subgroup X is definable by the following axioms:

1. $1 \in X$
2. $\forall x \in X \; x^{-1} \in X$
3. $\exists x_1, \ldots, x_k \; \forall x, y \in X \; \exists w \in X \; \bigvee_{i=1}^{k} xy = x_i w$

A first version of the Lie model theorem was proven in a pseudofinite setting by Hrushovski in [4] and this was the one that yielded the classification of finite approximate subgroups. This is a simplified statement of the theorem.

Theorem 0.7 (Lie Model Theorem)**.** *If X is an approximate subgroup, then there exists a subset Y of X^4 such that finitely many translates of Y cover X^4 and there is a homomorphism of groups from $\langle X \rangle$ to a connected Lie group with kernel in Y.*

Later, the result was extended to all approximate subgroups again by Hrushovski in [5]. In the general version, the group homeomorphism becomes a quasi-homomorphism.

Definition 0.8 (Quasi-homomorphism)**.** Let G be a semigroup, H a group, and C a small subset of H. The map $f : G \to H : C$ is a quasi-homomorphism with error set C if $f(xy) \in f(x)f(y)C \cup Cf(x)f(y)$.

The theorem roughly states that for every approximate subgroups there exists a generalized definable locally compact model.

Definition 0.9 (Generalized definable locally compact model)**.** Let G be a group and $X \subseteq G$ an approximate subgroup. A generalized definable locally compact model of X is a quasi-homomorphism $f : G \to H : C$ for some symmetric, normal, compact subset C of a locally compact group H such that:

1. for every compact $V \subseteq H$ there is $i \in \mathbb{N}$ such that $f^{-1}(V) \subseteq X^i$,
2. for every $i \in \mathbb{N}$, $\overline{f(X^i)}$ is compact in H,
3. there is $l \in \mathbb{N}$ such that for any compact Z, $Y \subseteq H$ with $C^l Y \cap C^l Z = \emptyset$ the preimages can be separated by a definable set.

Theorem 0.10 (Generalized Lie Model Theorem)**.** *There exists a generalized definable locally compact model of X with $l = 2$. Moreover, $f^{-1}(C) \subseteq X^{30}$ and there is a compact neighbourhood U of the neutral element in H such that $f^{-1}(U) \subseteq X^{14}$ and $f^{-1}(UC) \subseteq X^{34}$.*

In [6] Pillay and Krupinski showed an alternative proof of the theorem. They apply topological dynamics to the following context. Let X be an approximate subgroup definable in a structure M in a language $\mathcal{L}$ and $N \succ M$ an $|M|^+$-saturated elementary extension and $\mathcal{U} \succ N$ the monster model. $G = \langle X \rangle$ is a group and $G \subseteq M$. Let $S_{X^n,M}(N)$ be the space of complete types over N, containing the formula defining X^n, and finitely satisfiable in M. Denote by $S_{G,M}(N)$ the union $\bigcup_n S_{X^n,M}(N)$. Then this space with the relative topology inherited from the type space topology of $S_M(N)$ is a locally compact space on which we can define a product $*$.

Definition 0.11 (Product $*$)**.** Take $p, q \in S_{G,M}(N)$. Then $p * q = tp(ab/N)$ where $a \models p$, $b \models q$, and $tp(a/N, b)$ is finitely satisfiable in M.

Lemma 0.12. *The space $S_{G,M}(N)$ with the operation $*$ is a left topological semigroup*

We can then identify every $g \in G$ with $\mathrm{tp}(g/N)$ and define the action of G over $S_{G,M}(N)$ as $g * p$ for every $p \in S_{G,M}(N)$. Applying topological dynamics to the action $(G, S_{G,M}(N))$, it is then possible to find a subgroup M of $S_{G,M}(N)$ and a normal subgroup H of M such that M/H is the locally compact group needed in the theorem.

From the model theoretic point of view, it is interesting to consider the definability of the generalized definable locally compact model. The aim of this paper will indeed be to give an idea of in which sense it is definable. As we are working in the type space, we need the notion of hyperdefinable sets.

Definition 0.13 (Hyperdefinability)**.** Let $A \subseteq N$ be a set of parameters. A set P is A-hyperdefinable if it is of the form $P = X/E$ where X is type-definable over A and E is an equivalence relation type-definable over A.

We can also define a topology on hyperdefinable sets, the logic topology.

Definition 0.14 (Logic topology)**.** The logic topology of a hyperdefinable set is the quotient topology of the type space topology.

Notice that now if we identify a type $p \in S(N)$ with the equivalence class of any $a \models p$ modulo $\equiv_N$, the relation $\equiv_N$ is type-definable over N as $\bigwedge_{\varphi(x) \in \mathcal{L}(N)} (\varphi(x) \leftrightarrow \varphi(y))$ and this turns $S_{X^n,M}(N)$ into a hyperdefinable set for all $n \in \mathbb{N}$.

It seems like we found the right notion of definability for the generalized definable locally compact model. But there are two main problems regarding the locally compact group we are building. One is that it is a quotient in the type space, so a quotient of a hyperdefinable set. The other is that it is not closed in the logic topology but in the quotient topology of the relative topology on $S_{G,M}(N)$. Hence, in conclusion, we need another notion of hyperdefinability to answer the problem, which is what it is called relative hyperdefinability.

Definition 0.15 (Relative hyperdefinability)**.** A set C is relatively hyperdefinable if it is the quotient of a type-definable set intersected with the set of elements realizing $S_{G,M}(N)$ and a type-definable equivalence relation intersected with the set of elements realizing $S_{G,M}(N)$.

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On the role of connectivity in Linear Logic proofs

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1 Introduction

Linear logic (LL, [8]) is a refinement of both classical and intuitionistic logic in which a formula is treated as a computational resource, thanks to a decomposition of the connectives for conjunction ($\wedge$) and disjunction ($\vee$) into their multiplicative ($\otimes$ and $\parr$) and additive (& and $\oplus$) variants, and thanks to the exponential modalities that bridge the gap between them and make the structural rules of contraction and weakening explicit in the syntax (see Figure 1).

One of the main contributions of LL to the field of proof theory is the shift from a term-like or tree-like representation of proofs to a more general *graph*-like one, which removes much of the bureaucratic order imposed by sequent calculus and exposes the intrinsic graph structure of proofs. Graphs representing proofs in LL are usually called *proof-nets*, and are special inhabitants of a wider set of *proof-structures* which may not represent proofs. It is clearly an interesting theoretical question to find abstract properties (*correctness criteria*) allowing us to isolate, among proof-structures, those that are images of sequent calculus proofs (*sequentializable* proof-structures).

One of the most famous correctness criteria is the Danos-Regnier ACC property ([3]), relating the logical correctness of a proof in multiplicative LL without the units (MLL^-) with the acyclicity and connectivity of every graph obtained by detaching one premise for each $\parr$ node. However, as soon as one adds the multiplicative units to MLL^- (the fragment thus obtained is denoted by MLL), connectivity is lost: when translating the $\bot$ rule of sequent calculus in proof-nets, the $\bot$ node is isolated from the rest of the graph (Figure 3d). It is common practice ([16, 9, 1], ...) to add *jumps* to proof-structures to recover ACC, but this technique has major downsides (jumps have no canonical position in general, their behavior through cut elimination is a mess, ...).

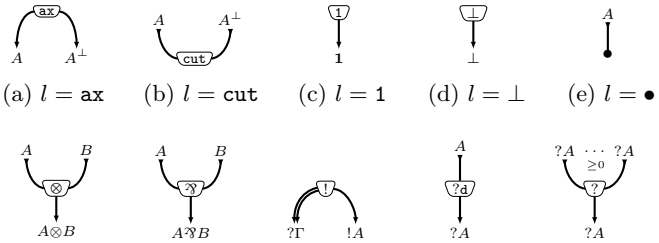
On the other hand, it is quite natural to extend ACC in the presence of the units by relating the number of connected components ($\sharp\mathsf{cc}$) and the number of $\bot$ nodes ($\sharp\mathsf{w}$) through the equality $\sharp\mathsf{cc} = \sharp\mathsf{w} + 1$ ([15]). Although this condition is not sufficient for a proof-structure of MLL to be sequentializable (Figure 4a), its extension to multiplicative exponential linear logic (MELL), that we denote by $\mathsf{ACC}_{\sharp\mathsf{w}}$, is necessary (Proposition 7) and stable under cut elimination (Theorem 9).

We then reverse the point of view by asking what the logical counterpart of connectivity is. More concretely, we identify a purely geometric restriction on proof-structures that turns $\mathsf{ACC}_{\sharp\mathsf{w}}$ into a sufficient condition for a proof-structure of MELL to be sequentializable (Theorem 11), and then turn this restriction into a fragment of MELL for which $\mathsf{ACC}_{\sharp\mathsf{w}}$ is a correctness criterion in the absence of cuts (Corollary 13). As a byproduct, we obtain that $\mathsf{ACC}_{\sharp\mathsf{w}}$ is a correctness criterion for the classical polarization of MELL in the absence of cuts (Corollary 14), and thus $\mathsf{ACC}_{\sharp\mathsf{w}}$ is equivalent in this setting to the criterion by Laurent ([11, 12, 13]). In the intuitionistic polarization of MELL (IMELL), $\mathsf{ACC}_{\sharp\mathsf{w}}$ is equivalent to the requirement of having exactly one output formula in the conclusion sequent (Proposition 18), which is equivalent to the familiar requirement of having exactly one formula on the right side of the turnstile in intuitionistic linear logic with two-sided sequents ([9]). Finally, we mention a result from a related work ([4]) showing that

$$\frac{}{\vdash A, A^\perp}\ ax \qquad \frac{\vdash \Gamma, A \quad \vdash A^\perp, \Delta}{\vdash \Gamma, \Delta}\ cut \qquad \frac{}{\vdash \mathbf{1}}\ \mathbf{1} \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \bot}\ \bot \qquad \frac{\vdash \Gamma, A \quad \vdash B, \Delta}{\vdash \Gamma, A \otimes B, \Delta}\ \otimes$$

$$\frac{\vdash \Gamma, A, B}{\vdash \Gamma, A \parr B}\ \parr \qquad \frac{\vdash ?\Gamma, A}{\vdash ?\Gamma, !A}\ ! \qquad \frac{\vdash \Gamma, A}{\vdash \Gamma, ?A}\ ?d \qquad \frac{\vdash \Gamma, ?A, ?A}{\vdash \Gamma, ?A}\ ?c \qquad \frac{\vdash \Gamma}{\vdash \Gamma, ?A}\ ?w$$

Figure 1: Sequent calculus rules.



(a) $l = \mathtt{ax}$ (b) $l = \mathtt{cut}$ (c) $l = \mathtt{1}$ (d) $l = \bot$ (e) $l = \bullet$

(f) $l = \otimes$ (g) $l = \parr$ (h) $l = !$ (i) $l = ?\mathtt{d}$ (j) $l = ?$

Figure 2: The local constraints that a node of a ground-structure with label l has to respect.

$\mathsf{ACC}_{\sharp \mathsf{w}}$ is a correctness criterion for the fragment of MELL that corresponds to the half-polarized typing system for call-by-push-value in [6, 7], in the absence of multiplicative cuts (Theorem 19).

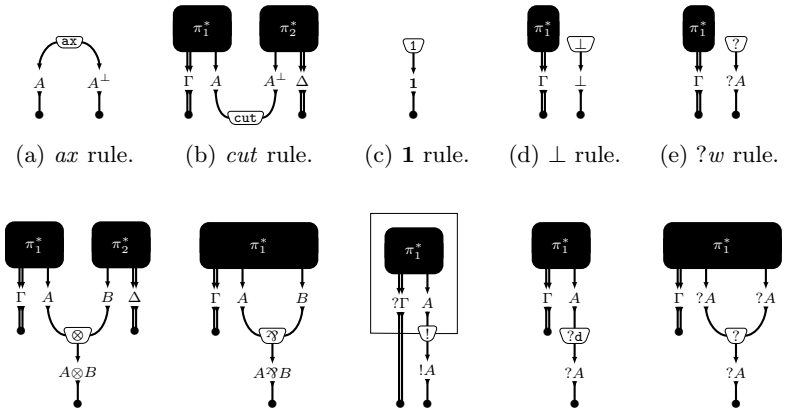
2 Proof-nets

Formulae of MELL are defined by the grammar: $A ::= X \mid \mathbf{1} \mid \bot \mid A \otimes A \mid A \parr A \mid !A \mid ?A$.

Definition 1. A *fragment* $\mathcal{F}$ of MELL is a set of formulae of MELL which is closed under sub-formulae, meaning that, for every $A \in \mathcal{F}$ and for every sub-formula B of A, we have $B \in \mathcal{F}$. A *sequent* of a fragment $\mathcal{F}$ of MELL is a finite multiset of formulae of $\mathcal{F}$. A *sequent calculus proof* in $\mathcal{F}$ with conclusion Γ is a finite tree of sequents of $\mathcal{F}$ obtained by using the rules of Figure 1.

Definition 2. A *ground-structure* G *of* MELL is a directed graph with nodes labeled by $\mathtt{ax}$, $\mathtt{cut}$, $\mathbf{1}$, $\bot$, $\otimes$, $\parr$, $!$, $?\mathtt{d}$, $?$ or $\bullet$, arcs labeled by formulae of MELL (called *types*), and such that every node satisfies the local constraints in Figure 2. If a is an arc from n to m, we say that a is a *conclusion* of n and a *premise* of m. A $?\mathtt{w}$ (resp. $?\mathtt{c}$) node is a $?$ node with no (resp. at least two) premises. The number of $\bot$ or $?\mathtt{w}$ nodes of G is denoted by $\sharp\mathsf{w}(G)$. Let $\mathcal{F}$ be a fragment of MELL. We say that G is a ground-structure of $\mathcal{F}$ if its arcs are labeled by formulae of $\mathcal{F}$. A *proof-structure of* $\mathcal{F}$ is a pair $R = (G_R, \mathsf{box}_R)$ with G_R a ground-structure of $\mathcal{F}$ and box_R a function mapping every $!$ node n of G_R to a proof-structure of $\mathcal{F}$ (a *box* of R) whose ground-structure has the same number of conclusions as n, and of the same types. Boxes are disjoint from the ground-structure G_R and pairwise. We write $R' \sqsubset R$ if R' is a box of R. A *conclusion* of R is a premise of a $\bullet$ node of G_R.

Definition 3. Let G be a ground-structure of MELL. A *switching* φ of G is a function mapping every $\parr$ or $?\mathtt{c}$ node to one of its premises. The *switching graph of* G *induced by* φ, denoted by G^φ, is the undirected graph obtained from G by adding a fresh $\bullet$ node n_0 for each premise a of a $\parr$ or $?\mathtt{c}$ node n such that $a \neq \varphi(n)$, and by setting a to be incident to n_0 rather than n. The number of connected components of G^φ is denoted by $\sharp\mathtt{cc}(G^\varphi)$.



(a) *ax* rule. (b) *cut* rule. (c) **1** rule. (d) ⊥ rule. (e) ?*w* rule.

(f) ⊗ rule. (g) ⅋ rule. (h) ! rule. (i) ?*d* rule. (j) ?*c* rule.

Figure 3: Desequentialization of sequent calculus proofs in MELL into proof-structures of MELL. A double arrow to a • node represents a finite number of arcs, each targeting a distinct • node.

The well-known process of desequentialization maps a sequent calculus proof π in MELL with conclusion $A_1, \dots, A_k$ to a proof-structure π^* of MELL with conclusions of types $A_1, \dots, A_k$. The function $\pi \mapsto \pi^*$ is defined by induction on π (see Figure 3). Full details are provided in [1].

Definition 4. Let $\mathcal{F}$ be a fragment of MELL. A proof-structure R of $\mathcal{F}$ is *sequentializable* when there exists a sequent calculus proof π in $\mathcal{F}$ such that $R = \pi^*$.

Definition 5. Let G be a ground-structure and let φ be a switching of G. We write:
$G^\varphi \models \mathsf{AC}$ if G^φ is acyclic; $G^\varphi \models \mathsf{C}$ if $\sharp\mathsf{cc}(G^\varphi) = 1$; $G^\varphi \models \mathsf{C}_{\sharp\mathsf{w}}$ if $\sharp\mathsf{cc}(G^\varphi) = \sharp\mathsf{w}(G) + 1$.
We write $G \models \mathsf{AC}$ (resp. C, $\mathsf{C}_{\sharp\mathsf{w}}$) when, for every switching φ of G, $G^\varphi \models \mathsf{AC}$ (resp. C, $\mathsf{C}_{\sharp\mathsf{w}}$).

Definition 6. Let R be a proof-structure. We write:

- $R \models \mathsf{AC}$ (resp. C, $\mathsf{C}_{\sharp\mathsf{w}}$) if $G_R \models \mathsf{AC}$ (resp. C, $\mathsf{C}_{\sharp\mathsf{w}}$) and, for $R_0 \sqsubset R$, $R_0 \models \mathsf{AC}$ (resp. C, $\mathsf{C}_{\sharp\mathsf{w}}$);
- $R \models \mathsf{ACC}$ if $R \models \mathsf{AC}$ and $R \models \mathsf{C}$; $R \models \mathsf{ACC}_{\sharp\mathsf{w}}$ if $R \models \mathsf{AC}$ and $R \models \mathsf{C}_{\sharp\mathsf{w}}$.

Proposition 7. *Let R be a proof-structure of* MELL. *If R is sequentializable, then $R \models \mathsf{ACC}_{\sharp\mathsf{w}}$.*

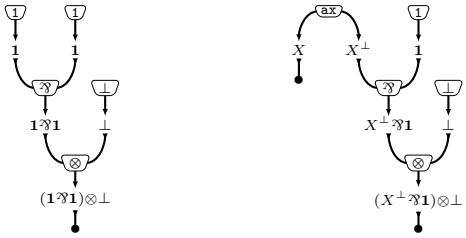
Remark 8. The converse of Proposition 7 does not hold, because of the example in Figure 4a, due to Laurent Regnier (common knowledge in the community, implicit in [15]).

Cut elimination on proof-structures of MELL is essentially a graph rewriting ([1]). We write $R \to R'$ if R reduces to R' in exactly one cut elimination step. A normal proof-structure of MELL is called *cut-free*. A notable property of $\mathsf{ACC}_{\sharp\mathsf{w}}$ is its stability under cut elimination ([10, 5, 1]):

Theorem 9. *Let R, R' be proof-structures of* MELL. *If $R \models \mathsf{ACC}_{\sharp\mathsf{w}}$, then $R' \models \mathsf{ACC}_{\sharp\mathsf{w}}$.*

3 Connectivity in general proof-nets

Definition 10. A ⊥, ⅋, ?d, ? or • node of a ground-structure G of MELL is *erasing* if its premises are all conclusions of erasing nodes. We say that G is a $(\neg\mathsf{w}\otimes)$-*ground-structure* when no premise of a cut or ⊗ node is a conclusion of an erasing node. A proof-structure R of MELL is a $(\neg\mathsf{w}\otimes)$-*proof-structure* if G_R is a $(\neg\mathsf{w}\otimes)$-ground-structure and every $R' \sqsubset R$ is a $(\neg\mathsf{w}\otimes)$-proof-structure.



(a) Classical (Regnier). (b) Intuitionistic (Section 3).

Figure 4: Two proof-structures of MELL satisfying $\mathsf{ACC}_{\sharp\mathsf{w}}$ which are not sequentializable.

Theorem 11 ([5])**.** *Let R be a $(\neg\mathsf{w}\otimes)$-proof-structure. If $R \models \mathsf{ACC}_{\sharp\mathsf{w}}$, then R is sequentializable.*

Definition 12. The fragment $(\neg?\otimes)\mathsf{LL}$ of MELL is defined by the following grammar:

$$A ::= X \mid \mathbf{1} \mid A \otimes A \mid A \parr A \mid A \parr E \mid E \parr A \mid !A \mid !E \qquad E ::= \bot \mid E \parr E \mid ?A \mid ?E$$

Corollary 13. *For R cut-free proof-structure of $(\neg?\otimes)\mathsf{LL}$: $R \models \mathsf{ACC}_{\sharp\mathsf{w}} \Leftrightarrow R$ is sequentializable.*

Hence, $\mathsf{ACC}_{\sharp\mathsf{w}}$ is a correctness criterion for the proof-structures of the classical polarization of MELL ([11, 14]) and, in particular, for the image of the call-by-name λ-calculus.

Corollary 14. *Let LL_{pol} be the fragment of MELL defined by the following grammar:*

$$P ::= \mathbf{1} \mid !X \mid !N \mid P \otimes P \quad \textit{(positives)} \qquad N ::= \bot \mid ?X \mid ?P \mid N \parr N \quad \textit{(negatives)}$$

For every cut-free proof-structure R of LL_{pol}, we have: $R \models \mathsf{ACC}_{\sharp\mathsf{w}} \Leftrightarrow R$ is sequentializable.

Remark 15. A refinement of Corollaries 13 and 14 yields canonical positions for jumps ([5]).

4 Connectivity in intuitionistic proof-nets

In this section, the set of atoms is split in two subsets: output atoms (X) and their duals ($X^\perp$).

Definition 16 ([2, 15, 9])**.** The fragment IMELL of MELL is defined by the following grammar:

$$O ::= X \mid \mathbf{1} \mid O \otimes O \mid O \parr I \mid I \parr O \mid !O \qquad \text{(outputs)}$$
$$I ::= X^\perp \mid \bot \mid I \parr I \mid I \otimes O \mid O \otimes I \mid ?I \qquad \text{(inputs)}$$

Proposition 17 ([9])**.** *Every provable sequent of IMELL has exactly one output formula.*

Proposition 18 ([5])**.** *Let R be a proof-structure of IMELL such that $R \models \mathsf{AC}$. Then $R \models \mathsf{ACC}_{\sharp\mathsf{w}}$ if and only if R has exactly one output conclusion.*

4.1 Further work and further restriction ([4])

Theorem 19. *Let VIMELL be the fragment of MELL defined by the grammar:*

$$P ::= X \mid \mathbf{1} \mid P \otimes P \mid !O \quad \textit{(positives)} \qquad N ::= X^\perp \mid \bot \mid N \parr N \mid ?I \quad \textit{(negatives)}$$
$$O ::= P \mid O \parr N \mid N \parr O \quad \textit{(outputs)} \qquad I ::= N \mid I \otimes P \mid P \otimes I \quad \textit{(inputs)}$$

For a multiplicative-normal proof-structure R of VIMELL: $R \models \mathsf{ACC}_{\sharp\mathsf{w}} \Leftrightarrow R$ is sequentializable.

Remark 20. Like in the classical polarization of linear logic, positive formulae carry a structure of !-coalgebra, meaning that they can be "equipped with structural rules", but we still keep the benefits of the intuitionistic polarization: VIMELL contains not just the call-by-name λ-calculus, but also the call-by-value λ-calculus and, more generally, the bang calculus.

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How to formulate and prove model completeness for the infinitary logic $\mathcal{L}_{\infty,\omega}$

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The infinitary logic $\mathcal{L}_{\infty,\omega}$ generalizes first order logic by allowing infinite disjunctions and conjunctions of sets $\{\phi_i(x_1,\dots,x_n) \mid i \in I\}$ of formulae on a fixed set of free variables $x_1,\dots,x_n$. Several concepts not expressible by first order formulae are axiomatizable by an $\mathcal{L}_{\infty,\omega}$-sentence, for example that of connected graph. However, a major drawback of the Tarski semantics for this logic is that it fails most of the cornerstone results holding in first order logic: completeness, compactness, and interpolation for $\mathcal{L}_{\infty,\omega}$ all fail for this semantics.

In recent work [5, 6, 7] Santiago Suárez and Viale, recovering a line of results which dates back to works by Karp [1], Makkai [3], Keisler [2], and Mansfield [4] among others, show through an approach rooted in the forcing method of Set Theory that natural reformulations for $\mathcal{L}_{\infty,\omega}$ of completeness, compactness, and interpolation can be proved if $\mathcal{L}_{\infty,\omega}$ is endowed with a boolean valued semantics. Furthermore, their work outlines a correspondence between forcing and consistency properties, a standard tool used to build models of $\mathcal{L}_{\infty,\omega}$-theories.

In this talk I will extend these results outlining that also the notion of model completeness can be elegantly phrased in the boolean valued semantics and, furthermore, that this formulation is strictly tied with the notion of genericity, which plays a central role in the forcing technique.

This is joint work with Matteo Viale.

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New results about reverse mathematics and wqo definitions

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1 Reverse mathematics and wqos

The term "reverse mathematics" refers to a broad research program in the foundations of mathematics, whose main goal is to find the minimal axioms needed to prove the theorems of ordinary mathematics. One topic of interest in reverse mathematics is the theory of *well-quasi-orders* ([Mar20]).

Definition 1.1. A quasi-order is a pair $(Q, \preceq)$ where Q is a set and $\preceq$ is a binary relation on Q which is reflexive and transitive.

The notion of well-quasi-order (or wqo for short) has many equivalent definitions.

Theorem 1.2. *Let $(X, \leq)$ be a quasi-order. The following are equivalent:*

(w1) for every infinite sequence $(x_n)_{n\in\omega}$ of elements of Q, there exist $n < m$ such that $x_n \preceq x_m$;

(w2) $(X, \leq)$ has no infinite descending sequences and no infinite antichains;

(w3) for every sequence $(x_n)_{n\in\omega}$ there is $A \subseteq \mathbb{N}$ infinite such that for every $n, m \in A$ with $n < m$ we have $x_n \leq x_m$;

(w4) for every $Y \subseteq X$ there is $Z \subseteq Y$ finite such that for every $y \in Y$ there is $z \in Z$ with $z \leq y$;

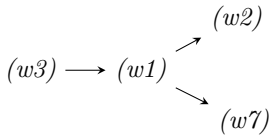
(w5) there is no infinite sequence of upward closed sets of X which is strictly increasing with respect to inclusion;

(w6) there is no infinite sequence of downward closed sets of X which is strictly decreasing with respect to inclusion;

(w7) every linear extension of $\leq$ is a well order.

In 2005, Cholak, Marcone and Solomon [CMS04] studied the computational strength of the equivalences contained in Theorem 1.2. Their work mostly concerns the distinction between definitions that are equivalent in the base theory of reverse mathematics RCA_0 (they turn out to be (w1), (w4), (w5) and (w6)), and those which are not. In particular, they proved the following result.

Theorem 1.3. *The implications between (w1), (w2), (w3) and (w7) which are provable in RCA_0 are exactly the ones in the transitive closure of the diagram:*



Our aim is revisiting that article and refining the results.

1.1 Work done

We improved Theorem 1.3 by adding two new definitions. The first one is the conjunction $(w2) \wedge (w7)$.

Theorem 1.4. $(w2) \wedge (w7) \to (w1)$ *follows from either* WKL_0 *or* ADS*, but it is not provable within* WWKL_0.

Another equivalent definition of wqo, which appears in [Die01] and we call (wD) is: "$\mathcal{P}(X)$ is well-founded with respect to $\preceq^\flat$", where $\preceq^\flat$ is defined by

$$X \preceq^\flat Y \iff \forall x \in X\, \exists y \in Y\ x \preceq y.$$

It is easy to see that (wD) implies $(w1)$ within RCA_0.

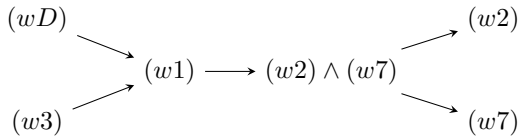
Theorem 1.5. *Over* RCA_0*, the implication* $(w1) \to (wD)$ *is equivalent to* ACA_0.

Theorem 1.6. *Over* RCA_0*, the implication* $(wD) \to (w3)$ *is equivalent to* $\mathsf{RT}^1_{<\infty}$.

Theorem 1.7. *Over* ADS*, the implication* $(w3) \to (wD)$ *is equivalent to* ACA_0.

These results are summarized in the following.

Corollary 1.8. *The implications between* (wD), $(w1)$, $(w2)$, $(w3)$, $(w7)$ *and* $(w2) \wedge (w7)$ *which are provable in* RCA_0 *are exactly the ones in the transitive closure of the diagram:*



A few questions are still open:

Question 1.9. *Are implications* $(w7) \to (w1)$ *and* $(w7) \to (w2)$ *equivalent to* WKL_0 *over* RCA_0*?*

Question 1.10. *Is the implication* $(w3) \to (wD)$ *equivalent to* ACA_0 *over* RCA_0*?*

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A Proof-Theoretic Treatment of Incorrect/Incomplete Proofs via Hilbert's Epsilon Calculus *

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Many significant mathematical discoveries initially came accompanied by incorrect or incomplete proofs, famously exemplified by Euler's first attempt at solving the Basel problem [5], [3]. Such historical instances prompt two central questions in mathematical proof theory:

1. If a proof is incomplete but intuitively reasonable, what is the minimal additional information required to complete and validate it?

2. If a proof is reasonable yet incorrect, how can we systematically identify a suitably weakened statement that can be correctly established through a closely related argument?

These questions are inherently dual: an incomplete proof of a suitably weakened statement, when completed, yields a valid proof of an implication towards the originally intended result.

The concept of weakest precondition plays a central role in identifying minimal assumptions required to validate a proof. Intuitively,

Definition 1. *A precondition for a satisfiable formula B is any formula A such that $A \to B$ is valid. We define an order among the satisfiable preconditions as $K < C$, meaning that $K \to C$ is valid but $C \to K$ is not valid. A formula A is a* weakest precondition *for B if:*

- *$A \to B$ is valid*
- *A is satisfiable*
- *A is <- minimal among other satisfiable preconditions.*

Thus the weakest preconditions contain the reasonable minimal assumptions under which B still holds.

We exclude $\bot$ (falsum) from being considered as a weakest precondition, as it trivially implies any formula, as $\bot \to B$ is always valid. Note that only $\bot$ has no preconditions. The order $<$ might not be linear; there can be multiple incomparable weakest preconditions. In the propositional setting, the weakest preconditions can be directly computed from truth tables.

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Every conjunction of entries representing a line of the truth table assigned $\top$ is a weakest precondition, as it is easily to be checked.

However, what to do with first-order expressions? To determine the minimal lacking information to make the expression valid is undecidable as "$\bot$ is the minimal information to make A valid" is equivalent to "A is valid". Furthermore, such a weakest information might not exist.

$$\rightarrow (\forall x(A(p(x)) \rightarrow A(x)) \rightarrow A(0)$$

has using Herbrand's theorem the valididating premises $A(0), A(p(0)), A(p(p(0))), \ldots$ and $A(p^{n+1}(0))$ is weaker then $A(p^n(0))$. Consequently, analysis in first-order logic requires examining specific incorrect proofs, often formulated within a sequent calculus framework.

Example 1. $(\forall x(A(x) \wedge \exists y B(y)) \rightarrow \forall x A(x)) \wedge \exists x C(x)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{A(a) \Rightarrow A(a)}{A(a) \wedge \exists y B(y) \Rightarrow A(a)}}{\forall x(A(x) \wedge \exists y B(y)) \Rightarrow A(a)} \quad \dfrac{\dfrac{\exists y B(y) \Rightarrow \exists y B(y)}{\exists y B(y) \Rightarrow \exists y C(y) \quad *}}{A(a) \wedge \exists y B(y) \Rightarrow \exists y C(y)}}{\dfrac{\forall x(A(x) \wedge \exists y B(y)) \Rightarrow \forall x A(x) \qquad \forall x(A(x) \wedge \exists y B(y)) \Rightarrow \exists y C(y)}{\forall x(A(x) \wedge \exists y B(y)) \Rightarrow \forall x A(x) \wedge \exists y C(y)}}$$
$$\dfrac{\forall x(A(x) \wedge \exists y B(y)) \Rightarrow \forall x A(x) \wedge \exists y C(y)}{\Rightarrow \forall x(A(x) \wedge \exists y B(y)) \rightarrow \forall x A(x) \wedge \exists y C(y)}$$

The obvious correction of the error is obtained from $\exists x B(x) \rightarrow \exists y C(y)$, but the argument is also valid if $\neg \forall x A(x)$ holds. $\neg \forall x A(x) \vee (\exists y B(y) \rightarrow \exists y C(y))$ is the guessed approach to a weakest premise. But how to calculate this in general?

To address these complexities, we employ Hilbert's epsilon calculus [2], [4], [1], a formalism based on the replacement of $\exists x A(x)$ by $A(\varepsilon_x A(x))$ and $\forall x A(x)$ by $A(\varepsilon_x \neg A(x))$, enabling first-order proofs to be analyzed in a manner similar to propositional logic. By systematically translating proofs into epsilon calculus, we can leverage methods akin to propositional analysis to ascertain weakest preconditions and correction strategies.

Example: The standard translation of $\exists x A(x) \wedge \forall x B(x)$ is $A(\varepsilon_x A(x)) \wedge B(\varepsilon_x \neg B(x))$.

LK-proofs are translated to ε-calculus by replacing $\exists$-right and $\forall$-left by $A(t) \rightarrow A(\varepsilon_x A(x))$ and $\neg A(t) \rightarrow \neg A(\varepsilon_x \neg A(x))$ on the left side of the sequent. $\exists$-left and $\forall$-right are replaced by substitution of the corresponding ε-terms and discharged. The resulting sequent is a tautology, which is an ε-proof in itself; therefore, it can be handled as in the propositional case.

In this talk we will systematically introduce and develop proof-theoretic tools based on ε-calculus for analyzing and correcting flawed mathematical proofs. Furthermore, we will discuss recent theoretical results demonstrating the robustness, or false-tolerance, of epsilon calculus, ensuring that even in cases of minimal errors, useful computational information can still be extracted from incorrect proofs.

As an example, we provide the following theorem.

Theorem: The algorithm of the extended first ε-theorem is false-tolerant: if there is at most only one interpretation that falsifies the proof (i.e., one line of the minimal truth table) then the same holds for the Herbrand disjunction obtained after elimination of critical formulas.

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A completeness theorem for topological doctrines

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As is well known, topological spaces provide a complete semantics for propositional S4 logic, where formulas are interpreted as subsets of a fixed space X and where the modality $\Diamond$ is interpreted as the closure operator. Following [Ghi90; GM91], we lift this semantics to *first-order* S4 logic in analogy with the classical set-based semantics of first-order Boolean logic. More precisely, in the mono-sorted case:

- Relational symbols of arity n are interpreted as subspaces of X^n, for a fixed space X.
- Functional symbols are interpreted as *continuous* maps.
- Boolean operations are interpreted as intersection, union and complementation.
- Existential quantification is interpreted by taking the direct image.
- Substitution of free variables is interpreted by taking the inverse image.

Compared to the case of first-order Boolean logic, the main subtlety is that substitution cannot be treated as a secondary construction on formulas: it is at the same level as the other logical connectives such as conjunction, disjunction, quantification and the modalities. The reason is that taking the inverse image by a continuous map does not commute with the modalities in topological spaces: we only have $\Diamond f^{-1}(A) \subseteq f^{-1}(\Diamond A)$ in general. Equality fails for instance when f is the diagonal $X \to X \times X$ of a non-discrete space X.

In order to make these ideas precise, we use the formalism of categorical logic. The base setup is developed in [GM25b]. In order to work with modal theories in an algebraic way, we define *modal categories*, which relate to first-order modal logic in the same way as Boolean categories relate to first-order Boolean logic. The main peculiarity is that in this setting, the bare categorical structure is not enough to specify a theory. Indeed, the notion of an "embedding" (subobject) differs from the notion of an "injection," as can be seen in the category of topological spaces, for instance. We use a proper factorization system $(\mathcal{E}, \mathcal{M})$ to encode this data: the "embeddings" are the morphisms in $\mathcal{M}$ while the "injections" are the monomorphisms.

A *lax morphism* of modal algebras is a map f preserving the Boolean operations and satisfying $\Diamond f(A) \leq f(\Diamond A)$.

Definition 1. A (Boolean) *modal category* is a left exact category E equipped with:

1. A stable proper factorization system $(\mathcal{E}, \mathcal{M})$.
2. For all $X \in \mathsf{E}$, a structure of modal algebra on the poset $\mathrm{Sub}_{\mathcal{M}}(X)$ of $\mathcal{M}$-subobjects of X, making $\mathrm{Sub}_{\mathcal{M}}$ a functor from E^{op} to the category of modal algebras and lax morphisms.

Moreover, we require that $\Diamond A = m^* \Diamond m A$ for each $m : X \hookrightarrow Y$ in $\mathcal{M}$ and each $A \in \mathrm{Sub}_{\mathcal{M}}(X)$ (we use m^* for inverse image along m and m itself for its left adjoint, namely direct image).

In [GM25a], we use this framework to state and answer to the following question:

What is the first-order modal theory of topological spaces?

We find two additional axioms, besides the obvious S4 axioms from the propositional case:

1. A "product independence axiom" which states that $\Diamond$ distributes over conjunctions of formulas sharing no common variable:

$$(\Diamond\varphi)[\overline{x}] \wedge (\Diamond\psi)[\overline{y}] \iff \Diamond(\varphi[\overline{x}] \wedge \psi[\overline{y}])$$

   (here notations like $\varphi[\overline{x}], \psi[\overline{y}]$ refer to inverse images along projections in internal logic).

2. A "loop contraction axiom," which expresses a continuity condition on certain composable loops of binary relations.

Theorem 1 ([GM25a])**.** *A small modal category embeds in a power of the modal category of topological spaces if and only if it satisfies S4 axioms, the product independence axiom and the loop contraction axiom.*

We give details about the loop contraction axiom. A *partial map* from X to Y is a morphism $f : D \to Y$ defined on an $\mathcal{M}$-subobject D of X. Its graph is an internal binary relation $G_f \subseteq X \times Y$ whose transpose we denote by $G_f^t \subseteq Y \times X$. A *loop* is a sequence of internal binary relations $R_1, R_2, \ldots, R_n$ such that the composite $R_1 \circ \cdots \circ R_n$ makes sense and such that the codomain of R_1 coincides with the domain of R_n. This object is called the *anchor* of the loop. We define the *acceptable loops* by induction:

1. The empty loop is acceptable (on any anchor).
2. Identities can be added anywhere in an acceptable loop to produce a new acceptable loop.
3. If w and w' are acceptable loops, then w, w' is acceptable.
4. If w is acceptable and if f is a partial map, then G_f, w, G_f^t is acceptable.
5. If $R_1, \ldots, R_n$ and $S_1, \ldots, S_n$ are acceptable, then $R_1 \times S_1, \ldots, R_n \times S_n$ is acceptable.

The loop contraction axiom states that for any acceptable loop $R_1, \ldots, R_n$ and for any subspace A of the anchor, it holds that

$$\Diamond R_1 \Diamond R_2 \Diamond \cdots \Diamond R_n A \leq \Diamond A.$$

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Partially-ordered Płonka sums of po-algebras

José Gil-Férez, Peter Jipsen, and Giuseppe Zecchini

Abstract

Płonka sums have played a significant role in universal algebra and logic, e.g., with the construction of Clifford semigroups from groups, involutive bisemilattices from Boolean algebras, regularization of varieties, and its connection to logics of variable inclusion [3]. More recently they have been applied to the structure theory of commutative idempotent involutive residuated lattices [6], locally integral involutive po-semigroups [5], and residuated po-monoids [2]. Here we present a general theory for partially-ordered Płonka sums of residuated structures within the setting of a certain kind of partially-ordered algebras (see [8] for details on po-algebras; see [4] or [7] for details on residuated structures). We also define Płonka partitions for posets and explain how they are related to semilattice-indexed families of pairs of maps ψ_{ij}, φ_{ij} that define po-Płonka sums. The main application of our results is to residuated lattices, but the framework presented here applies to n-ary residuated partially ordered algebras in general. Even in the setting with no fundamental operations or only unary operations, po-Płonka sums provide a new tool for constructing or decomposing posets and lattices with pairs of operations that are residuals, Galois connections or dual Galois connections.

1 Po-Płonka sums

A *po-algebra signature* is a tuple $(\mathcal{F}, \tau)$, where $\mathcal{F}$ is a set of (functional) symbols and τ is a map $\mathcal{F} \to \{\vee, \wedge\} \times \bigcup_{n=0}^{\infty} \{1, \partial\}^n$. A *po-algebra* $\mathbf{A}$ in the po-signature $(\mathcal{F}, \tau)$ is a t-uple $(A, F, \leqslant^{\mathbf{A}})$, where $(A, \leqslant^{\mathbf{A}})$ is a non-empty poset and $F = \{f^{\mathbf{A}}\}_{f \in \mathcal{F}}$ is a set of operations on A indexed by $\mathcal{F}$ and such that for every $f \in \mathcal{F}$:

1. $f^{\mathbf{A}}$ is an $Ar(f)$-ary operation on A, where $Ar(f)$ is the length of the second component of $\tau(f)$, called, as usual, *arity* of f;
2. for every $i \in \{1, \dots, n\}$, $\tau(f)_i$ denotes the tonicity of the i-th variable, i.e. $f^{\mathbf{A}}$ is monotone in the i-th variable if $\tau(f)_i = 1$ and antitone if $\tau(f)_i = \partial$;
3. let $\circ \in \{\vee, \wedge\}$ be the first component of $\tau(f)$ and $\mathbf{0} \in \{\bot, \top\}$. For every $i \in \{1, \dots, n\}$, let $\circ_i$ and $\mathbf{0}_i$ be defined, respectively, as $\circ$ and $\mathbf{0}$ if $\tau(f)_i = 1$ and their duals otherwise. If the required joins/meets exists, then

$$f^{\mathbf{A}}(x_1, \dots, x \circ_i y, \dots, x_n) = f^{\mathbf{A}}(x_1, \dots, x, \dots, x_n) \circ f^{\mathbf{A}}(x_1, \dots, y, \dots, x_n)$$

   If the required extremes exists, then

$$f^{\mathbf{A}}(x_1, \dots, \mathbf{0}_i, \dots, x_n) = \mathbf{0}$$

For each $f \in \mathcal{F}$, the operation $f^{\mathbf{A}}$ is said to be $\vee$-*tonic* if $\tau(f)_0 = \vee$, and $\wedge$-tonic if $\tau(f)_0 = \wedge$.

A *po-metamorphism* between two po-algebras over the same po-signature $(\mathcal{F}, \tau)$ is a pair of monotone maps $\psi, \varphi : \mathbf{A} \to \mathbf{B}$ such that $\psi(x) \leqslant^{\mathbf{B}} \varphi(x)$ and for all $f, g \in \mathcal{F}$ such that $\tau(f)_0 = \vee$ and $\tau(g)_0 = \wedge$ we have

$$\varphi(f^{\mathbf{A}}(x_1, \dots, x_n)) = f^{\mathbf{B}}(\varphi^{\tau_1}(x_1), \dots, \varphi^{\tau_n}(x_n)), \qquad \text{where } \tau_i = \tau(f)_i$$
$$\psi(g^{\mathbf{A}}(x_1, \dots, x_n)) = g^{\mathbf{B}}(\psi^{\tau_1}(x_1), \dots, \psi^{\tau_n}(x_n)), \qquad \text{where } \tau_i = \tau(g)_i,$$

$\varphi^\partial = \psi = \psi^1$ and $\psi^\partial = \varphi = \varphi^1$.

Note that if a metamorphism ψ, φ satisfies $\varphi = \psi$ then it is a po-algebra homomorphism. Po-metamorphisms form a category, denoted by $\mathsf{Alg}^{\leqslant}_\tau$, where $id_{\mathbf{A}}, id_{\mathbf{A}}$ is the identity po-metamorphism on $\mathbf{A}$.

A *semilattice directed system of po-algebras and metamorphisms* is a functor $\Phi\colon \mathbf{I} \to \mathsf{Alg}^{\leqslant}_\tau$. Let $\psi_{ij}, \varphi_{ij} = \Phi(i \leqslant j)\colon \mathbf{A}_i \to \mathbf{A}_j$. The *po-Płonka sum* $\mathbf{A} = \int_{\mathbf{I}} \Phi$ is defined by endowing the disjoint union $A = \uplus_I A_i$ with the relation

$$a \leqslant^{\mathbf{A}} b \iff \varphi_{i_1 j}(a) \leqslant^{\mathbf{A}_j} \psi_{i_2 j}(b), \qquad \text{for } j = i_1 \vee i_2 \text{ in } \mathbf{I} \text{ and } a \in A_{i_1}, b \in A_{i_2},$$

and for $k = i_1 \vee \cdots \vee i_n \in I$ and $a_j \in A_{i_j}$, the fundamental operations for all $f, g \in \mathcal{F}$ such that $\tau(f)_0 = \vee$ and $\tau(g)_0 = \wedge$ are defined by

$$\begin{aligned} f^{\mathbf{A}}(a_1, \dots, a_n) &= f^{\mathbf{A}_k}(\varphi^{\tau_1}_{i_1 k}(a_1), \dots, \varphi^{\tau_n}_{i_n k}(a_n)), && \text{where } \tau_i = \tau(f)_i \\ g^{\mathbf{A}}(a_1, \dots, a_n) &= g^{\mathbf{A}_k}(\psi^{\tau_1}_{i_1 k}(a_1), \dots, \psi^{\tau_n}_{i_n k}(a_n)), && \text{where } \tau_i = \tau(g)_i. \end{aligned}$$

We also need the following properties for all $i < j, k \in I$ and $l = j \vee k$:

(S1) $\varphi_{jl} \circ \psi_{ij} \leqslant \psi_{kl} \circ \varphi_{ik}$,

(S2) $\varphi_{ij}(x) \leqslant \psi_{ij}(y) \implies x < y$, for all $x, y \in A_i$.

Theorem 1. *Let Φ be a directed system of po-metamorphisms with po-Płonka sum $\mathbf{A}$.*

1. *The binary relation $\leqslant^{\mathbf{A}}$ is a partial order if and only if (S_1) and (S_2) hold.*
2. *If for all $i \in I$, an operation $f^{\mathbf{A}_i}$ has a residual $g_j^{\mathbf{A}_i}$ with respect to each argument x_j, then $f^{\mathbf{A}}$ has residual $g_j^{\mathbf{A}}$ in the po-Płonka sum, i.e., the following equivalence is preserved*

$$f(x_1, \dots, x_j, \dots, x_n) \leqslant y \iff x_j \leqslant g_j(x_1, \dots, y, \dots, x_n)$$

where $\tau(f)_0 = \vee$, $\tau(g_j)_0 = \wedge$.

2 Lattices from step-by-step po-Płonka sums

It is clear from the definition of po-algebra over a po-signature that it is not possible to exploit the formalism described in the previous section to treat directly lattices as algebraic structures; therefore, a different approach is required for the preservation of lattice operations. In particular, we employ a *step-by-step* approach. We begin characterizing the two-element Płonka sums of join-semilattices.

Proposition 2. *A join-semilattice $\mathbf{I}$ is a Płonka sum over a semilattice direct system $\mathbb{I} = (\{\mathbf{I}_1, \mathbf{I}_2\}, \{1 < 2\}, \varphi_{12})$ of disjoint join-semilattices if and only if $\mathbf{I}_1$ is an ideal of $\mathbf{I}$ and $\mathbf{I}_2$ is a retract of $\mathbf{I}$ whose universe is an order filter of $\mathbf{I}$.*

Notably, therefore, a decomposition into a two-fiber Płonka sum of a join-semilattice $\mathbf{I}$ is always possible in the presence of a splitting of $\mathbf{I}$. Even more specifically, if $\mathbf{I}$ is a *finite lattice* (that is, a finite join-semilattice with zero), then the correspondence between two-fiber Płonka sum decompositions of $\mathbf{I}$ and splittings of $\mathbf{I}$ is one-to-one. Thus, in the case of finite lattices, two-fiber Płonka sum decompositions are closely related to so-called *interval dismantlable lattices* (see [1]). In view of this, and inspired by the conditions in Proposition 2, we provide the following definition.

Definition 1. We say that a join-semilattice $\mathbf{I}$ is *Płonka 2-dismantable* if there exists a well-founded tree of join-semilattices rooted at $\mathbf{I}$ such that for every non-leaf node $\mathbf{J}$ in the tree:

1. the children of $\mathbf{J}$ are two join-semilattices $\mathbf{J}_1$ and $\mathbf{J}_2$ whose universes $\{J_1, J_2\}$ form a partition of J, where J_1 is an ideal and J_2 is an order filter of $\mathbf{J}$;
2. $\mathbf{J}_2$ is a retract of $\mathbf{J}$;
3. every leaf of the tree is a trivial join-semilattice.

By Proposition 2, a join-semilattice is Płonka dismantable if and only if it can be recursively decomposed into two-fiber Płonka sums until the trivial components are reached. This recursive decomposition provides the theoretical framework for our *step-by-step (po-)Płonka sums*. To ensure the process is well-defined, we first generalize [9, Theorem 3.2] to po-Płonka sums.

Theorem 3 (Associativity for po-Płonka sums)**.** *Let $\Phi : \mathbf{I} \to \mathsf{Alg}^{\leqslant}_{\tau}$ be a directed system of po-Algebras satisfying $(S_1), (S_2)$, where $\Phi(i) = \mathbf{A}_i$ for every $i \in I$, and $\theta : \mathbf{J} \to \mathsf{Jslat}$ a directed system of join-semilattices, where $\theta(j) = \mathbf{I}_j$ for every $j \in J$, whose Płonka sum is $\mathbf{I}$. Define the directed system $\Phi' : \mathbf{J} \to \mathsf{Alg}^{\leqslant}_{\tau}$, where $\Phi'(j) = \int_{\mathbf{I}_j} \Phi$, while for every $j, k \in J$ with $j \leqslant k$ let*

$$\varphi'_{jk} := \bigcup_{i \in I_j} \varphi_{i\theta_{jk}(i)} \quad \text{and} \quad \psi'_{jk} := \bigcup_{i \in I_j} \psi_{i\theta_{jk}(i)},$$

then Φ' satisfies $(S_1), (S_2)$ and $\int_{\mathbf{I}} \Phi = \int_{\mathbf{J}} \Phi'$.

Then we identify conditions under which the poset po-Płonka sum of two lattices is again a lattice extending the component lattices. Finally, applying Theorem 2 we show that po-Płonka sums over Płonka dismantable join-semilattices can be constructed in a step-by-step manner by using only two lattices at each step. With this approach, well-behaved po-Płonka sums of lattice-ordered algebras are again lattice ordered.

3 Poset po-Płonka sums from poset partitions

Here we address the deconstruction process of a poset under the framework of poset po-Płonka sums.

Definition 2. A *poset partition* is a tuple $(P, \leqslant, \pi)$, where $(P, \leqslant)$ is a poset and $\pi = \{P_i\}_{i \in I}$ is a partition of P. For $i \in I$, the induced order on each P_i is denoted $\leqslant_i$. For every $i, j \in I$ define the partial maps $\varphi_{ij}, \psi_{ij} : P_i \to P_j$ as follows

$$\varphi_{ij}(x) = \min(\uparrow x \cap P_j) \qquad \text{and} \qquad \psi_{ij}(x) = \max(\downarrow x \cap P_j), \qquad \text{for every } x \in P_i. \tag{1}$$

Moreover, let $\leqslant_{\mathbf{I}}$ be the binary relation on $\mathbf{I}$ defined, for every $i, j \in I$, by

$$i \leqslant_{\mathbf{I}} j \text{ iff } \varphi_{ij},\ \psi_{ij} \text{ are both total maps.} \tag{2}$$

Proposition 4. *$(I, \leqslant_{\mathbf{I}})$ is a poset such that $\varphi_{ik} = \varphi_{jk} \circ \varphi_{ij}$ and $\psi_{ik} = \psi_{jk} \circ \psi_{ij}$ for every $i, j, k \in I$ such that $i \leqslant_{\mathbf{I}} j \leqslant_{\mathbf{I}} k$ if and only if the following two conditions hold:*

(I_1) *For every $i, j, k \in I$ and for every $x \in P_i$ and $z \in P_k$, if $i \leqslant_{\mathbf{I}} j \leqslant_{\mathbf{I}} k$ and $x \leqslant z$, then there exists $y \in P_j$ such that $x \leqslant y \leqslant z$;*

(I_2) *For every $i, j, k \in I$ and for every $x \in P_i$ and $z \in P_k$, if $i \leqslant_{\mathbf{I}} j \leqslant_{\mathbf{I}} k$ and $z \leqslant x$, then there exists $y \in P_j$ such that $z \leqslant y \leqslant x$.*

Theorem 5. *Let $(P, \leqslant, \pi)$ be a poset partition satisfying (I_1), (I_2) and suppose the index poset $\mathbf{I} = (I, \leqslant_{\mathbf{I}})$ is a join-semilattice. Then π with the family of maps φ and ψ as in (1) is a semilattice directed system of posets and metamorphisms and $(P, \leqslant)$ is its poset po-Płonka sums if and only if the following condition holds:*

(I_3) *For every $i, j \in I$ and $x \in P_i, y \in P_j$, if $x \leqslant y$ then there exists $\ell \geq i, j$ and $z \in P_\ell$ such that $x \leqslant z \leqslant y$.*

Finally, suppose we consider a (join-)semilattice direct system of poset and metamorphisms $(\{P_i\}_{i\in I}, (I, \leqslant^I), \{\widehat{\varphi_{ij}}, \widehat{\psi_{ij}}\}_{i\leqslant^I j})$; then we are naturally left with a poset partition $(P, \leqslant, \{P_i\}_{i\in I})$, where $(P, \leqslant)$ is the poset po-Płonka sum of the system, from which we can obtain families of (partial) maps φ and ψ as in (1), and a binary relation $\leqslant_I$ on I as in (2). We wonder under which conditions these two constructions are inverses of each other.

Theorem 6. *Let $(\{P_i\}_{i\in I}, (I, \leqslant^I), \{\widehat{\varphi}_{ij}, \widehat{\psi}_{ij}\}_{i\leqslant^I j})$ be a semilattice directed system of posets and metamorphisms and $(P, \leqslant)$ its poset po-Płonka sum. The following are equivalent:*

(i) $\leqslant_I \subseteq \leqslant^I$;

(ii) $(P, \leqslant, \{P_i\}_{i\in I})$ *satisfies* (I_1);

(iii) $(P, \leqslant, \{P_i\}_{i\in I})$ *satisfies* (I_2).

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Free Nelson algebras dually

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Constructive logic with strong negation, also known as *Nelson logic* or simply as *N3*, was developed by Vorob'ev [12] to formalize the idea, due to Nelson [8] and Markov [7], that refuting a formula by constructing a counterexample yields a negation that is different from the intuitionistic one. Formally, this logic extends the intuitionistic propositional calculus (IPC) by adding a new unary connective $\sim$, called *strong negation*, and the following axioms:

$$(\sim(\varphi \wedge \psi)) \leftrightarrow (\sim\varphi \vee \sim\psi) \qquad (\sim\sim\varphi) \leftrightarrow \varphi$$
$$(\sim(\varphi \vee \psi)) \leftrightarrow (\sim\varphi \wedge \sim\psi) \qquad (\sim\varphi) \rightarrow (\varphi \rightarrow \psi)$$
$$(\sim(\varphi \rightarrow \psi)) \leftrightarrow (\varphi \wedge \sim\psi)$$

Białynicki-Birula and Rasiowa [2] proved that this logic is a *conservative extension* of IPC. Therefore, it shares the constructive nature of intuitionistic logic but enriches it with a stronger, symmetric form of refutation.

In [9], Rasiowa introduced the algebraic structures now known as *Nelson algebras*, originally under the name of *quasi-pseudo-Boolean algebras*, and showed that they provide the algebraic semantics for constructive logic with strong negation.

Definition 1. *A* Nelson algebra *is an algebraic structure* $(N, \wedge, \vee, \rightarrow, \sim, 1, 0)$ *consisting of a bounded distributive lattice* $(N, \wedge, \vee, 1, 0)$ *equipped with a binary operation* $\rightarrow$ *and a unary operation* $\sim$ *satisfying the following equations:*

$$\sim\sim x = x \qquad x \rightarrow x = 1$$
$$\sim(x \vee y) = \sim x \wedge \sim y \qquad x \wedge (x \rightarrow y) = x \wedge (\sim x \vee y)$$
$$x \wedge \sim x \leq y \vee \sim y \qquad x \rightarrow (y \wedge z) = (x \rightarrow y) \wedge (x \rightarrow z)$$
$$x \rightarrow (y \rightarrow z) = (x \wedge y) \rightarrow z$$

The class of Nelson algebras form a variety; i.e., an equational class. Spinks and Veroff [11] proved that the variety of Nelson algebras is term-equivalent to a variety of bounded, commutative, and integral residuated lattices, and hence constructive logic with strong negation can be thought of as a substructural logic (see also [3] for a simpler algebraic proof).

Definition 2. *A Nelson algebra* N *is said to be* free *over* $X \subseteq N$ *if for every Nelson algebra* A *and function* $f\colon X \to A$*, there exists a unique homomorphism* $g\colon N \to A$ *extending* f*.*

Free Nelson algebras are, up to isomorphism, the *Lindenbaum-Tarski algebras* for constructive logic with strong negation. Indeed, let X be a set of propositional variables and $\mathsf{Form}(X)$ the set of formulas built over X in the language of constructive logic with strong negation. Define an equivalence relation $\sim$ on $\mathsf{Form}(X)$ by setting $\varphi \sim \psi$ iff $(\varphi \leftrightarrow \psi) \wedge (\sim\varphi \leftrightarrow \sim\psi)$ is a theorem of the logic. The quotient $\mathsf{Form}(X)/\!\sim$ has naturally a structure of Nelson algebra, which is free over (the set of equivalence classes of the elements of) X. Moreover, if a Nelson

algebra N is free over X, then it is isomorphic to $\mathsf{Form}(X)/\sim$. Therefore, free Nelson algebras provide substantial information about constructive logic with strong negation.

Our goal is to use duality to obtain a description of free Nelson algebras exploiting the connection between Nelson algebras and Heyting algebras, which are the structures providing the algebraic semantics of IPC. Recall that a *Heyting algebra* $(H, \wedge, \vee, \to, 1, 0)$ consists of a distributive lattice $(H, \wedge, \vee, 1, 0)$ equipped with a binary operation $\to$ such that $a \leq b \to c$ iff $a \wedge b \leq c$ for all $a, b, c \in H$.

Sendlewski [10] unveiled a tight link between Nelson algebras and Heyting algebras by proving that the category NA of Nelson algebras and homomorphisms between them is equivalent to the category HF defined as follows.

Definition 3. *The objects of* HF *are pairs* (H, F)*, where* H *is a Heyting algebra and* F *is a filter of* H *such that* H/F *is a Boolean algebra. A morphism* $f\colon (H_1, F_1) \to (H_2, F_2)$ *in* HF *is a Heyting homomorphism* $f\colon H_1 \to H_2$ *such that* $f[F_1] \subseteq F_2$*.*

We exploit this equivalence to obtain a duality for Nelson algebras. Recall that a subset of a topological space is *clopen* if it is closed and open at the same time. A *Stone space* is a compact Hausdorff topological space in which every open is a union of clopen subsets. Moreover, a subset U of a poset $(X, \leq)$ is called an *upset* if $x \in U$ and $x \leq y$ implies $y \in U$, for every $y \in X$.

Definition 4.

1. *A* Priestley space $(X, \leq)$ *consists of a Stone space* X *equipped with a partial order* $\leq$ *such that for all* $x, y \in X$ *with* $x \not\leq y$ *there exists a clopen upset* U *such that* $x \in U$ *and* $y \notin U$*.*
2. *An* Esakia space *is a Priestley space* $(X, \leq)$ *such that* $\{x \in X \mid x \leq y \text{ for some } y \in V\}$ *is clopen for every clopen subset* V *of* X*.*

Let Pries be the category of Priestley spaces and continuous order-preserving maps between them and let Esa be the subcategory of Pries consisting of Esakia spaces and continuous p-morphisms, where an order-preserving map $f\colon X \to Y$ between posets is a *p-morphism* if $f(x) \leq y$ implies that there is $z \geq x$ such that $f(z) = y$.

The clopen upsets of a Priestley space (resp. Esakia space) ordered by inclusion form a bounded distributive lattice (resp. Heyting algebra). These correspondences extend to the well-known Priestley and Esakia dualities (see, e.g., [6, 5]).

Theorem 5 (Priestley and Esakia dualities)**.**

1. Pries *is dually equivalent to the category of bounded distributive lattices and homomorphisms between them.*
2. Esa *is dually equivalent to the category of Heyting algebras and Heyting homomorphisms.*

We will work with the following analogs of Pries and Esa.

Definition 6.

1. *Let* PriesC *be the category whose objects are pairs* (X, C) *where* X *is a Priestley space and* C *is a closed subset of maximal points of* X *and whose morphisms are of the form* $f\colon (X_1, C_1) \to (X_2, C_2)$ *consisting of a* Pries*-morphism* $f\colon X_1 \to X_2$ *such that* $f[C_1] \subseteq C_2$*.*
2. *Let* EsaC *be the subcategory of* PriesC *whose objects are pairs* $(X, C) \in \mathsf{PriesC}$ *where* X *is an Esakia space, and whose morphisms* $f\colon (X_1, C_1) \to (X_2, C_2)$ *are* PriesC*-morphisms with* $f\colon X_1 \to X_2$ *an* Esa*-morphism.*

As an easy consequence of Esakia duality and the equivalence between NA and HF due to Sendlewski, we obtain a duality for Nelson algebras.

Proposition 7. EsaC *is dually equivalent to* NA.

Our goal is to describe the objects of EsaC that correspond to free Nelson algebras. It is well-known that the forgetful functor from the category of Heyting algebras to the category of bounded distributive lattices has a left adjoint. So, it follows from Priestley and Esakia dualities that the inclusion functor $\mathsf{Esa} \hookrightarrow \mathsf{Pries}$ has a right adjoint $\mathsf{R}\colon \mathsf{Pries} \to \mathsf{Esa}$. We proved that this adjunction extends to PriesC and EsaC.

Theorem 8. *The inclusion functor* $\mathsf{EsaC} \hookrightarrow \mathsf{PriesC}$ *has a right adjoint* $\mathfrak{R}\colon \mathsf{PriesC} \to \mathsf{EsaC}$, *which maps* $(X, C) \in \mathsf{PriesC}$ *to* $(\mathsf{R}(X), \varepsilon_X^{-1}[C]) \in \mathsf{EsaC}$, *where* ε *is the counit of the adjunction between the inclusion functor* $\mathsf{Esa} \hookrightarrow \mathsf{Pries}$ *and* $\mathsf{R}\colon \mathsf{Pries} \to \mathsf{Esa}$.

To every Priestley space we associate an object of PriesC whose components can be thought of as the subdiagonal and diagonal of X.

Definition 9. *Let* X *be a Priestley space. Define*

$$\mathsf{S}(X) = \{(x, y) \in X \times X \mid x \leq y\} \quad \textit{and} \quad \mathsf{D}(X) = \{(x, y) \in X \times X \mid x = y\},$$

and equip $\mathsf{S}(X)$ *with the topology induced by the product topology on* $X \times X$ *and the order given by* $(x_1, y_1) \leq (x_2, y_2)$ *iff* $x_1 \leq x_2$ *and* $y_2 \leq y_1$.

Proposition 10. *If* X *is a Priestley space, then* $(\mathsf{S}(X), \mathsf{D}(X)) \in \mathsf{PriesC}$.

Let 2 be the Priestley space consisting of two elements $0, 1$ with $0 < 1$ equipped with the discrete topology. Let X be a set. It is well known that 2^X equipped with the product topology and componentwise order is the Priestley dual of the bounded distributive lattice free over X (see, e.g., [6, Prop. 4.8]). The following is our main result providing a dual description of free Nelson algebras.

Theorem 11. *Let* X *be a set. Then* $\mathfrak{R}(\mathsf{S}(2^X), \mathsf{D}(2^X)) \in \mathsf{EsaC}$ *is dual to the Nelson algebra free over* X.

The result above can be adapted to many subvarieties of the variety of Nelson algebras. As an example, we show how it yields a dual description of free nilpotent minimum algebras generalizing the one obtained in [1, Sec. 4.3] to any number of free generators.

Nilpotent minimum algebras form a variety of bounded residuated lattices providing the algebraic semantics of the substructural propositional logic called *nilpotent minimum logic*.

Theorem 12. *[3, pp. 783–784] The variety of nilpotent minimum algebras is term-equivalent to the variety of Nelson algebras validating the equation* $(x \to y) \vee (y \to x) = 1$.

In view of Theorem 12, we denote by NM the full subcategory of NA consisting of the Nelson algebras validating $(x \to y) \vee (y \to x) = 1$. Let ERS denote the full subcategory of Esa consisting of *Esakia root systems*, which are the Esakia spaces X such that $\{y \in X \mid x \leq y\}$ is a linearly ordered subposet of X for every $x \in X$. As a consequence of [3, Thm. 6.19] and Proposition 7, we obtain a duality for NM reminiscent of the one established in [1].

Proposition 13. NM *is dually equivalent to the full subcategory* ERSC *of* EsaC *whose objects are pairs* $(X, C) \in \mathsf{EsaC}$ *with* $X \in \mathsf{ERS}$.

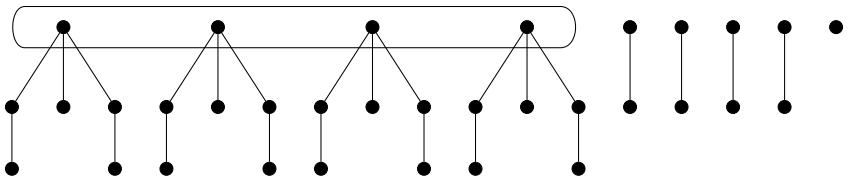
It is shown in [4] that the right adjoint $\mathsf{CC}\colon \mathsf{Pries} \to \mathsf{ERS}$ of the inclusion $\mathsf{ERS} \hookrightarrow \mathsf{Pries}$ sends $X \in \mathsf{Pries}$ to the set of all closed chains of X (i.e., nonempty closed subsets of X that are linearly ordered with respect to the order on X) suitably ordered and topologized.

Theorem 14. *The inclusion functor* $\mathsf{ERSC} \hookrightarrow \mathsf{PriesC}$ *has a right adjoint* $\mathfrak{CC}\colon \mathsf{PriesC} \to \mathsf{ERSC}$*, which maps* $(X, C) \in \mathsf{PriesC}$ *to* $(\mathsf{CC}(X), \hat{C}) \in \mathsf{ERSC}$*, where* $\hat{C} = \{\{x\} \mid x \in C\}$*.*

We obtain the following description of free nilpotent minimum algebras.

Theorem 15. *Let* X *be a set. Then* $\mathfrak{CC}(\mathsf{S}(2^X), \mathsf{D}(2^X)) \in \mathsf{ERSC}$ *is dual to the algebra of* NM *free over* X*.*

The diagram below depicts the object $(X, C) \in \mathsf{ERSC}$ dual to the free 2-generated algebra in NM, where the circled set of points represents C.



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The Theory of Meaningfulness in Untyped lambda-Calculi*

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One of the main goals of programming language theory is the rigorous mathematical logic study of the *meaning* of programming languages. The *untyped* λ*-calculus* [Chu41] is a simple and Turing-complete model of computation that represents the kernel of any functional programming language. There are actually many variants of the λ-calculus, mimicking different parameter passing styles in programming languages such as *call-by-name* and *call-by-value*, each one with its own theory and notion of meaningfulness. Via the Curry-Howard correspondence between programs and proofs, Girard's linear logic [Gir87] inspired the introduction of variants of the λ-calculus, such us the *bang calculus* [EG16, GM18], subsuming both call-by-name and call-by-value thanks to Girard's two translations of intuitionistic logic into linear logic.

The study of the semantics of the untyped *call-by-name* λ-calculus (CbN) is a well-developed field built around the concept of *solvable* terms, which are elegantly characterized in many different ways [Wad76, Bar77, CDC78, Bar84]. In particular, unsolvable terms provide a *consistent* notion of meaningless terms (in that they can all be equated without collapsing the induced equational theory), and meaningful terms can be identified with the solvable ones.

The study of the semantics of the untyped *call-by-value* λ-calculus (CbV, which is closer to the real implementations of programming languages [Plo75]) is instead still in its early stages, because of some inherent difficulties but also because CbV solvable terms are less studied and understood than in CbN. On the one hand, we show that a carefully crafted presentation of CbV [AP12, CG14], inspired by linear logic, allows us to recover many of the properties that solvability has in CbN, in particular logical (via multi types) and operational (via a special reduction) characterizations. On the other hand, we stress that, in CbV, solvability plays a different role: identifying unsolvable terms as meaningless induces an inconsistent equational theory actually identifying all terms [AG22].

We argue that in CbV, the correct notion of meaningful terms is captured by the concept of *potential valuability* [PR99]. In particular, terms that are not potentially valuable enjoy the same good properties as unsolvable terms in CbN:

1. they can be characterized logically via multi types [AG22];
2. they can be characterized operationally as non-termination of a special reduction [AP12];
3. they fulfill the *genericity lemma* according to which a term that is not potentially valuable plugged into a term whose evaluation terminates plays no role in that evaluation [AGK24];
4. they provide a *consistent* notion of meaningless terms in CbV (in that they can all be equated without collapsing the induced equational theory) [AGK24].

. Moreover, we can subsume the notions of meaningfulness for both CbN and CbV in a general notion of meaningfulness for the *bang calculus* [AGK24b]. Finally, we show that the study of meaningfulness in the bang calculus sounds more intricate than in CbN and CbV λ-calculi, because the bang calculus is a more expressive language.

*This contribution collects some results that the author obtained in collaboration with Alberto Carraro, Simona Ronchi Della Rocca, Luca Paolini, Beniamino Accattoli, Giulio Manzonetto, Delia Kesner, Victor Arrial, and others.

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▸ GIORGIO LAGUZZI, *Laver trees in the generalised Baire spaces.*
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This talk is based on the results published in [1]. We prove that any suitable generalization of Laver forcing to the space κ^κ, for uncountable regular κ, necessarily adds a Cohen κ-real. We also study a dichotomy and an ideal naturally related to generalized Laver forcing. Using this dichotomy, we prove the following stronger result: if $\kappa^{<\kappa} = \kappa$, then every $< \kappa$-distributive tree forcing on κ^κ adding a dominating κ-real which is the image of the generic under a continuous function in the ground model, adds a Cohen κ-real. This is a contribution to the study of generalized Baire spaces and answers a question from [2].

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Generators of varieties of ℓ-modules over subrings of $\mathbb{R}$

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A classical result in the theory of lattice-ordered structures (see [2, Theorem 6.1] and [6, Theorem 2.5]) states that the variety of Riesz spaces is, as a quasi-variety, generated by the ordered field $\mathbb{R}$ of real numbers. Noting that Riesz spaces are, by definition, ℓ-modules over $\mathbb{R}$, this talk addresses the problem of determining, for a unital subring $A \subseteq \mathbb{R}$, whether A generates the variety $A\ell\mathsf{Mod}$ of ℓ-modules over A as a quasi-variety.

The overarching objective is to transfer the universal-algebraic constructions known for Riesz spaces to the setting of ℓ-modules over A, and in particular to show that these modules lie in the class $\mathrm{SPP_u}(A)$, where the operators S, P, and $\mathrm{P_u}$ denote the formation of subalgebras, direct products, and ultraproducts, respectively. To this end, we adapt the strategy developed in [6] by establishing a series of embedding results for an arbitrary ℓ-module M over A.

First, *M embeds into a direct product of linearly ordered A-modules.* To justify this, we recall the notion of *f-module*, originally introduced in [3, 8]. An *f-module* is an ℓ-module that is isomorphic to a subdirect product of a family $\{M_i\}_{i\in I}$ of linearly ordered modules. Since 1 is a strong unit of A, every ℓ-module M over A is in fact an f-module (see [7, Lemma 6.2]), which yields the desired embedding into a product of linearly ordered components.

Next, *each M_i embeds into an ultraproduct of its finitely generated subalgebras.* This is an instance of a standard model-theoretic result (see [4, Theorem 2.14 of V]). The subsequent technical steps are devoted to proving that *any finitely generated linearly ordered A-module embeds into a subalgebra of $\mathbb{R}^n_{\mathrm{lex}}$ for some $n \in \omega$*, and then embedding $\mathbb{R}^n_{\mathrm{lex}}$ into an ultrapower of A. Since [6, Lemma 2.4] establishes that each $\mathbb{R}^n_{\mathrm{lex}}$ embeds into an ultrapower of $\mathbb{R}$, the remaining task reduces to *embedding $\mathbb{R}$ into an ultrapower of A.* This yields the following chain of inclusions, culminating in the desired conclusion that $A\ell\mathsf{Mod} = \mathrm{SPP_u}(A)$:

$$A\ell\mathsf{Mod} \subseteq \mathrm{SPSP_uSSP_uSP_u}(A) \subseteq \mathrm{SPP_u}(A) \subseteq A\ell\mathsf{Mod}.$$

To handle the embedding of finitely generated chains, it is necessary to take into account that, in contrast to the situation for Riesz spaces, the structure A is only assumed to be a ring. We resolve this by using the localisation of modules (see [1, Chapter 3]). Let N be a finitely generated linearly ordered module over a subring A of $\mathbb{R}$. We extend scalars to the field of fractions K of A by passing to the localised K-module $(A \setminus \{0\})^{-1}N$. We endow this localisation with a linear order by stipulating that

$$\frac{x}{a} \leq \frac{y}{b} \iff ab(ay - bx) \geq 0,$$

and we then show that the resulting linearly ordered K-vector space can be embedded into $\mathbb{R}^n_{\mathrm{lex}}$ for some $n \in \omega$. Finally, N embeds into $(A \setminus \{0\})^{-1}N$ via the $A\ell$-monomorphism $x \mapsto x/1$.

Theorem 1. *Let N be a finitely generated linearly ordered module over a subring A of $\mathbb{R}$. Then there exists $n \in \omega$ such that N admits an embedding into $\mathbb{R}^n_{\mathrm{lex}}$.*

To complete the composition of class operators, we consider the diagram $\mathrm{Diag}(\mathbb{R})$. Up to logical equivalence, any finite subset of $\mathrm{Diag}(\mathbb{R})$ can be reduced to a system consisting of A-linear equations $\ell_i(x) = 0$ and strict inequalities $\ell_j(x) > 0$. Since $\mathbb{R}$ is a model of this system, its solution set is given by the intersection of a linear subspace $V \subseteq \mathbb{R}^n$ with a nonempty open cone $C \subseteq \mathbb{R}^n$. Because the field of fractions K of A is dense in $\mathbb{R}$, the intersection $V \cap C$ contains a point $q \in K^n$. By clearing denominators, we thereby obtain a solution lying in A^n. The Compactness Theorem then ensures the existence of a corresponding model, which can be realised as an ultraproduct, and the method of diagrams (see [5, Proposition 2.1.8]) furnishes the desired final embedding.

Theorem 2. *If A is a subring of $\mathbb{R}$, then $\mathbb{R}$ embeds (as an $A\ell$-module) into an ultrapower of A.*

By assembling these intermediate embedding results, we obtain the main theorem of this work, thereby extending the classical generation theorem for Riesz spaces to the setting of ℓ-modules over subrings of $\mathbb{R}$.

Theorem 3. *If A is a subring of $\mathbb{R}$, then A generates the variety $A\ell\mathsf{Mod}$ of ℓ-modules over A as a quasi-variety.*

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From MV-algebras to non-Archimedean random variables AILA 2026

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Abstract

We explore the possibility of non-Archimedean random variables, valued in more general ranges than $[0, 1]$ and thus belonging to non-Archimedean MV-algebras. In particular, we considered two classes of MV-algebras: perfect MV-algebras and MV-algebras belonging to the Chang MV-algebra variety C, the simplest non-Archimedean MV-algebra

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1 Introduction

The classical theory of probability is traditionally grounded on set theoretic structures: events are subsets of a sample space and random variables are measurable functions defined on that space. Despite its effectiveness, this approach is not conceptually unavoidable. In his celebrated Fubini Lectures, Gian Carlo Rota raised what he called the first problem of probability: to reformulate probability theory in a point free way, replacing spaces of points with suitable algebras of random variables.

This perspective naturally leads to algebraic structures capable of encoding uncertainty independently of an underlying sample space. Within this framework, MV-algebras play a central role. Originally introduced by Chang as the algebraic semantics of Łukasiewicz many-valued logic, MV-algebras provide a natural environment for fuzzy events and fuzzy random variables, where truth values range over the real unit interval [0,1].

A crucial distinction within the theory of MV-algebras lies between Archimedean and non-Archimedean structures. Here an MV-algebra is called Archimedean if there is no infinitesimal. For instance Chang MV-algebra C is non-Archimedean, see [2]. (it turns out that Archimedean MV algebras coincide with semisimple MV algebras in the sense of universal algebra).

While Archimedean MV-algebras admit faithful representations as algebras of real-valued functions, non-Archimedean MV-algebras contain infinitesimal elements and cannot be represented in this classical way. However, this procedure raises a fundamental question: what structural information is lost when infinitesimals are ignored? In the following we will try to answer to this question.

2 Compact MV-spaces with perfect stalks

The duality theory for MV-algebras can be naturally formulated in terms of sheaf representations. Following Dubuc and Poveda, an MV-space is a Hausdorff topological space equipped with a sheaf of MV-algebras. Restricting this correspondence to the variety V(C), one obtains a more specific duality in which the stalks are perfect MV-algebras. We recall below the relevant definitions and the corresponding equivalence theorems.

Definition 1. *[6] Let X be a Hausdorff topological space and F a sheaf of MV-algebras over X. The pair (X,F) will be called an MV-space. For any $x \in X$, denote by F_x the stalk of F at the point x.*

Now $K_x := \{a \in F(X) | a_x = 0\}$ is an ideal in F(X), the MV-algebra of global sections of F. If U is an open subset of X, then we write $Ker\, U := \cap\{K_x | x \notin U\}$.

In [6], a definition of compact MV-space is given:

Definition 2. *An MV-space (X, F) is compact if the topological space X is compact and $K_x \oplus Ker\, U = F(X)$ for any $x \in X$ and any open neighborhood U of x.*
(X,F) is called local if every stalk is a local MV-algebra.

Theorem 1. *[6] The category of MV-algebras is equivalent to the category of compact MV-spaces with local stalks.*

Restricting MV-algebras to the variety $V(C)$, we obtain:

Theorem 2. *The category $V(C)$ is equivalent to the category of compact MV-spaces with perfect stalks.*

3 Boolean products and weak Boolean products

The representation of algebras in V(C) as weak Boolean products provides a useful structural description in terms of perfect MV-algebras. The following theorem establishes the general representation result, while the subsequent examples illustrate the distinction between strong and weak Boolean products in different situations.

Theorem 3. *Any nontrivial V(C)-algebra A is representable as a Weak Boolean Product, over the Stone space of $B(A)$, of perfect MV-algebras.*

Proof. See [3, Proposition 9.8] and [4, Theorem 5.9]. □

Let us make some examples:

Proposition 1.
1. *Let B be a Boolean algebra. Then the product of [6] on B is strong Boolean.*
2. *Let $Free_1$ be the free MV-algebra on one element. Then the Boolean product of [6] on $Free_1$ is weak but not strong.*
3. *There is an algebra A in $V(C)$ where the Boolean product of [6] is weak but not strong.*

4 States on perfect MV-algebras

Here is an example of definition of state on perfect MV-algebras, that is, generated by its infinitesimals:

Definition 3. *Let A be a perfect MV-algebra. A lex-state of A is a function S from A to the lexicographic product of groups $\mathbb{Z} \times_{lex} \mathbb{R}$ such that:*

1. $S(0) = (0, 0)$
2. *for every x in $Rad(A)$, $S(x) \in \{0\} \times (0, \mathbb{R}^+)$ and $S(\neg x) = (1, 0) - S(x)$.*
3. *For every $x, y \in Rad(A)$, $S(x \oplus y) = S(x) + S(y)$.*

Example 1. *Let C be the Chang's MV-algebra. Then an example of lex-state s is the immersion of C into $\mathbb{Z} \times_{lex} \mathbb{R}$.*

Another definition of state on perfect MV-algebras is the following:

Definition 4. *A Δ-state on a perfect MV-algebra $A = \Delta(G)$ is a map obtained by applying the Δ functor (introduced in [5]) to a state $s : G \to \mathbb{R}$.*

Note that a Δ-state is a map from A to $\Delta(\mathbb{R})$.

We claim that Definitions 3 and 4 collapse:

Proposition 2. *Lex-states and Δ-states coincide.*

5 Algebras of non-Archimedean random variables

In order to identify a suitable class of random-variable algebras within the variety V(C), we consider MV-algebras of continuous functions taking values in $\Delta(\mathbb{R})$. Two natural classes arise: quasi-constant functions and locally quasi-constant functions.

- The first class given by MV-algebras of quasi-constant functions denoted by $QC(X, \Delta(\mathbb{R}))$. That is, we have continuous functions from a compact Hausdorff space X to $\Delta(\mathbb{R})$ that take all values in the radical or all values in the co-radical of $\Delta(\mathbb{R})$;
- the second class is given by MV-algebras of functions called *locally-quasi-constant*, $LQC(X, \Delta(\mathbb{R}))$. These are continuous functions f from a compact Hausdorff space X to $\Delta(\mathbb{R})$ such that for every point y of X there is an open neighborhood $\mathcal{O}$ of y where f is quasi-constant in that the values of the function are infinitely close to each other, that is for every $a, b \in \mathcal{O}$, $d(f(a), f(b)) \in Rad(\Delta(\mathbb{R}))$.

In the case of $V(C)$-algebras, we propose, as random-variable algebras, MV-algebras of the form $LQC(X, \Delta(\mathbb{R}))$ of **locally** quasi-constant functions. Also we will see the role played by algebras of quasi-constant functions in describing properties of algebras of locally quasi-constant functions.

We claim that LQC provides an appropriate category of random variable algebras in V(C), as evidenced by the following proposition:

Proposition 3. *1. The category LQC is equivalent to UCC^*, the category of unitary commutative C^*-algebras (hence to the category of norm complete Riesz MV-algebras, see [7], theorem 4.5.1)*

2. *The Boolean part of* $LQC(X, \Delta(\mathbb{R}))$ *is* $C(X, \{0, 1\})$*, where* $C(X, \{0, 1\})$ *is the Boolean Algebra of continuous functions from* X *in* $\{0, 1\}$*.*

3. *The perfect part of* $LQC(X, \Delta(\mathbb{R}))$ *is* $QC(X, \Delta(\mathbb{R}))$*.*

4. *The category LQC is a full subcategory of* $V(C)$*.*

5. *Every MV-algebra in the category LQC has a maximal space which is a Stone space (i.e., a compact Hausdorff totally disconnected space).*

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Probabilistic Inference via Linear Logic Proofs

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Abstract

We propose a notion of *probabilistic proof* for the multiplicative and additive fragment of linear logic (MALL) formulated in graphical (*proof-nets*) and sequential style (*derivations*). The definition of probabilistic proof is proved to be correct and complete with respect to the notion of *Bayesian network* that is a graphical model for representing joint probability distributions over a set of discrete (Boolean) random variables.

Analogously to what happens with Bayesian networks, probabilistic proofs of linear logic are both an intelligible way of representing probabilities and an efficient method to reason about them.

1 Introduction

Linear logic [5] is a (constructive) refinement of classical and intuitionistic logic. Instead of emphasizing *truth*, as in classical logic, linear logic emphasizes the role of formulas as *resources* primarily played within *proofs*. Indeed, one of the most innovative contributions of linear logic is the introduction of two complementary syntaxes for proofs: a sequential one, for expressing proofs as derivation trees in the sequent calculus (*à la* Gentzen, [4]) and a parallel one, for representing proofs as more general graphs, also known as *proof nets*. Given its focus on resources and (mainly) graphical proofs, linear logic has found many applications in Computer Science where programs are theoretically interpreted as "graphs".

In this wake, we present here a notion of *probabilistic proof* of linear logic (both in *proof nets* and *sequential* style) that allows us to express efficiently both the graphical representation of probability distributions and the reasoning on them, similarly to what happens with Bayesian networks.

Bayesian networks [14, 3] are elegant and efficient graphical representations of probability distributions over random (discrete) variables. They are widely used in Artificial Intelligence and Machine Learning for different purposes such as automatic text classification, spam detection, sentiment analysis, supporting medical diagnosis, etc. Bayesian networks are very useful for inferring (or predicting) the probability of some events given other observed events (evidence). For this reason, it appears natural to try to embed Bayesian networks in logical systems in which it is possible to formalize reasoning on probabilities. Integration of Bayesian networks and first-order logic have been widely investigated (see, e.g. [15]) although it seems quite natural to express Bayesian networks in classical propositional logic.

Our proposal is to remain in the propositional fragment of linear logic trying to express efficiently the *graphical representation of probability distributions* via the *proof nets syntax* and, the *reasoning on them*, via the *sequent proofs syntax*. Formally, we set a correspondence between (Boolean) Bayesian networks and a special subclass of proof nets of the multiplicative-additive fragment (MALL, [1, 9, 8, 10, 11, 6]) with the Mix rule of the propositional fragment of linear logic: this is called the class of *Bayesian proof nets*. Proof nets are a graphical syntax that allows demonstrations of linear logic to be expressed in a modular and parallel way, abstracting away from the often useless bureaucracy of the sequential calculus.

Actually, it is not a novelty to propose variants of proof nets inspired by the methodologies used in Machine Learning or more generally by AI as, e.g., the neural variant of proof nets based on Sinkhorn networks by [7]. The novelty of Bayesian proof nets is given by the fact that they are special proofs endowed with a probability distribution over the set of their "slices". Intuitively, a multiplicative-additive proof net π with conclusions $A_1, ..., A_n$ can be viewed as a "sum" (a superposition) of simpler, multiplicative proof nets, called "slices". Therefore our idea is to associate a probability distribution to the set of slices associated to the proof net π; indeed, since each slice satisfies the invariance of having the same conclusions as π, we can see the probability distribution associated with the slices, as a probability distribution over the conclusions of π itself. Actually, slices play a double role:

- on one hand they allow to detect the "correctness" of the given MALL proof net;
- on the other hand (similarly to what the *chain rule of probabilities* does) they allow to factorize the joint probability distribution associated to the given proof net by means of the (conditional) probability distribution associated to the Bayesian links which the given proof net is built on. The basic idea is that Bayesian links capture the behavior of the discrete random variables of a Bayesian network.

It is already known [14, 3] that any Bayesian network $\mathcal{N}$ expresses the joint probability $Pr(A_1, ..., A_n)$ over its n random (discrete) variables $A_1, ..., A_n$. Here [2, 13] we show that any BN $\mathcal{N}$ can be encoded both by:

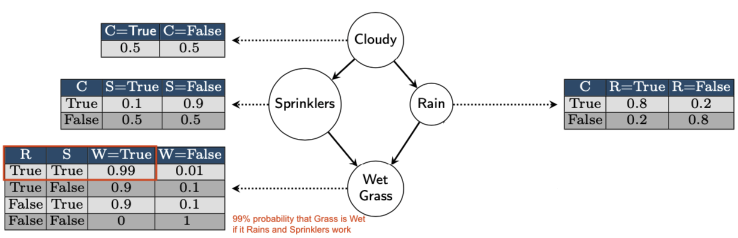
1. a **probabilistic proof net** $\pi^{\mathcal{N}}$ built over n special Bayesian links of MALL: exactly one Bayesian link α_i encodes a random variable/node A_i of $\mathcal{N}$; thus, each additive slice $S(\pi^{\mathcal{N}})$ (consisting of an appropriate mutilation of each B-link of $\pi^{\mathcal{N}}$) corresponds to the factorization of the joint probability $Pr(A_1, ..., A_n)$ over the conclusions, $A_1, ..., A_n$, of the proof net $\pi^{\mathcal{N}}$ [12];
2. and by a **focused probabilistic proof** $\Pi^{\mathcal{N}}$ of MALL with Mix rule [2]

$$\vdash \underbrace{p}_{\text{probability}} :: \underbrace{\mathcal{A}_1, ..., \mathcal{A}_n}_{\text{probabilitsic bipoles}} , \underbrace{a_1, ..., a_n}_{\text{atomic contest}}$$

where $\mathcal{A}_1, ..., \mathcal{A}_n$ are special formulas of MALL, called *bipoles* endowed with a probability distribution, and $a_1, ..., a_n$ are literals encoding the values of the random variables of the Bayesian Network.

The choice of linear logic as the underlying logical framework is motivated by its inherent resource sensitivity, which aligns well with the intuition that we should resolve the conditional probability of each node of Bayesian network only once per computation.

We give an overview of our approach through an example. Let us briefly recall some basic notions about them. A Bayesian network is a graphical model that represents a set of random variables and their conditional dependencies via a directed acyclic graph, where each node represents a random variable, and the directed edges represent the conditional dependencies between these variables. The one shown below, for example, is a Boolean Bayesian network with only four nodes/variables.



Each row of the *conditional probability table* associated to a node specifies the *conditional probability* $\mathsf{Pr}(\mathsf{X} \mid \mathsf{Y}_1, \ldots, \mathsf{Y}_m)$ that the variable X takes a specific value, given some *evidences*, i.e., given that the parent variables $\mathsf{Y}_1, \ldots, \mathsf{Y}_m$ take specific values. For example, with reference to the Bayesian network shown in the previous figure, the probability that the grass is wet, given the evidences that it is raining and the sprinklers are on, is 99%. In the general case, conditional probabilities for random variables $\mathsf{X}_1, \ldots, \mathsf{X}_n$, given evidences random variables $\mathsf{Y}_1, \ldots, \mathsf{Y}_m$, are defined by the following law, known as Bayes' Theorem,

$$\mathsf{Pr}(\mathsf{X}_1, \ldots, \mathsf{X}_n \mid \mathsf{Y}_1, \ldots, \mathsf{Y}_m) = \frac{\mathsf{Pr}(\mathsf{X}_1, \ldots, \mathsf{X}_n, \mathsf{Y}_1, \ldots, \mathsf{Y}_m)}{\mathsf{Pr}(\mathsf{Y}_1, \ldots, \mathsf{Y}_m)} \tag{1}$$

where $\mathsf{Pr}(\mathsf{X}_1, \ldots, \mathsf{X}_n, \mathsf{Y}_1, \ldots, \mathsf{Y}_m)$ is called the *joint probability* of $\mathsf{X}_1, ..., \mathsf{X}_n, \mathsf{Y}_1, ..., \mathsf{Y}_m$, and $\mathsf{Pr}(\mathsf{Y}_1, \ldots, \mathsf{Y}_m)$ is the *marginal probability* that $\mathsf{Y}_1, \ldots, \mathsf{Y}_m$ happen, regardless of the values of $\mathsf{X}_1, \ldots, \mathsf{X}_n$.

Our driving intuition is to to represent conditional probabilities by multi-head "methods" Equation (2) or equivalently by "bipoles" Equation (3) of linear logic thus, as in conditional probabilities, we can interpret the outcomes $\mathsf{X}_1, \ldots, \mathsf{X}_n$ as being *produced* simultaneously given the evidences $\mathsf{Y}_1, \ldots, \mathsf{Y}_m$:

$$\underbrace{\mathsf{Pr}(\mathsf{X}_1, \ldots, \mathsf{X}_n \mid \mathsf{Y}_1, \ldots, \mathsf{Y}_m) = p}_{\text{conditional probability}} \quad \rightsquigarrow \quad \underbrace{p}_{\text{probability}} \; :: \; \underbrace{[X_1, \ldots, X_n]}_{\text{multi-head}} \; :\!- \; \underbrace{[Y_1, \ldots, Y_m]}_{\texttt{body}} \tag{2}$$

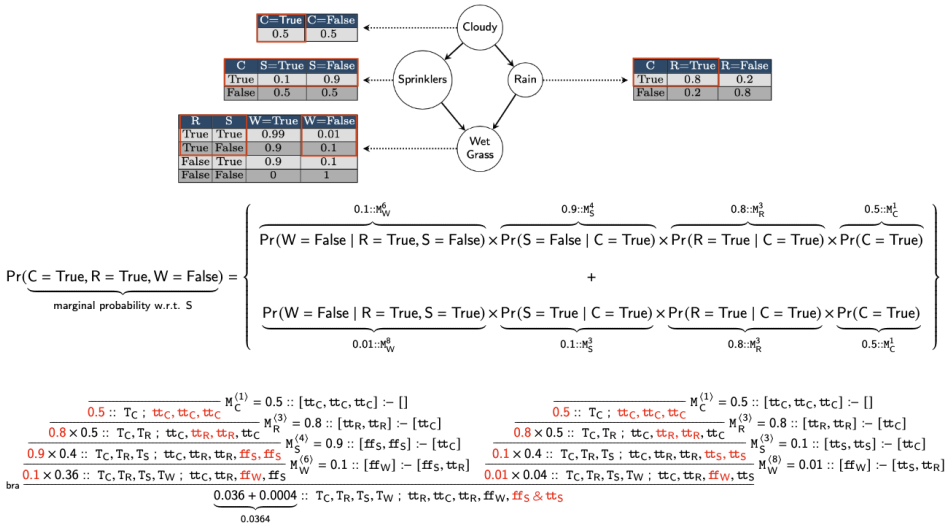
$$\overbrace{\underbrace{[h_{X_1}, \ldots, h_{X_n}]}_{\texttt{multi-head}} \; :\!- \; \underbrace{[b_{Y_1}, \ldots, b_{Y_m}]}_{\texttt{body}}}^{\texttt{method}} \quad \rightsquigarrow \quad \overbrace{\underbrace{(h^{\perp}_{X_1} \otimes \cdots \otimes h^{\perp}_{X_n})}_{\text{multi-head}} \otimes \underbrace{(b_{Y_1} \parr \cdots \parr b_{Y_m})}_{\text{body}}}^{\texttt{bipole}} \tag{3}$$

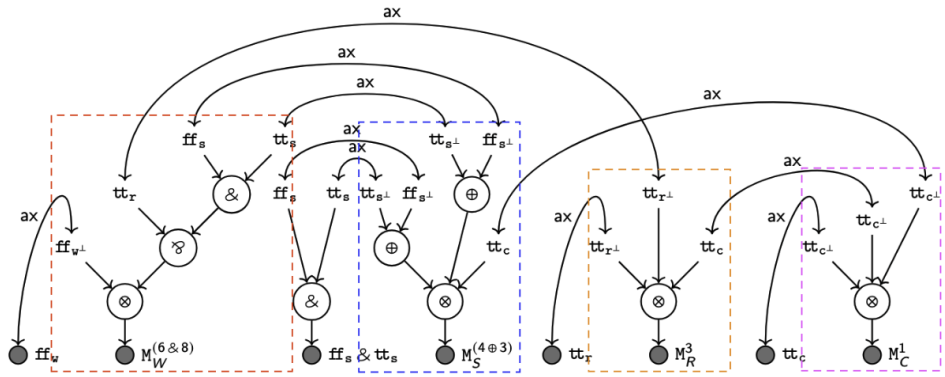
The effect of the probabilistic annotation is to modify the operational semantics of the method application in such a way that, when the method is applied during a proof search (*expansion* phase), the probability of the conclusion $q \cdot p$ is obtained by multiplying the probability of the premises q by the probability p of the method, as shown in Equation (4).

$$\mathsf{exp}\,\frac{}{\mathsf{P}, \mathtt{T} \; ; \; \Gamma, h_1, \ldots, h_n} \qquad \text{where } p :: [\mathtt{h_1}, \ldots, \mathtt{h_n}] \; :\!- \; [\mathtt{b_1}, \ldots, \mathtt{b_m}] \in \mathtt{T} \tag{4}$$

As application, we show how to represent Bayesian networks, and how to compute joint and marginal probabilities directly within proofs of linear logic without relying on an external semantic interpretation. More precisely, we show how proofs of linear logic naturally mimic the process of computing over a Bayesian network where probabilities, annotating the current state, are updated according to rules of the operational semantics, each corresponding to the application of a conditional probability defined in the network.

For example, we illustrate how the arithmetic expression of a possible way to compute the marginal probability $\mathsf{Pr}(\mathsf{C} = \mathsf{True}, \mathsf{R} = \mathsf{True}, \mathsf{W} = \mathsf{False})$, can be derived in linear logic, both by means of a sequent calculus proof and by means of a proof net. In the derivation, each application of a method $\mathtt{T_X}$ corresponds to the multiplication of the probability of the premises by the probability in the row of the conditional probability table of X corresponding to the values of the evidences in the premises; the branching step corresponds to the sum in the arithmetic expression.





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Probability of Quotient-like Conditionals

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A conditional statement is a hypothetical proposition of the form

"If [antecedent] is the case, then [consequent] is the case",

where the antecedent is assumed to be true.

The most well-known approach in formalizing conditionals comes from the philosophical perspective developed first by Stalnaker [10, 11] and then further analyzed by Lewis [6], that is grounded on particular Kripke-like structures. In particular, Lewis defines a hierarchy of logics for conditionals, which have been shown to be algebraizable in [9] with respect to varieties of Boolean algebras with operators, named *Lewis variably strict conditional algebras*.

One of the most natural questions that arise from the study of conditionals is their probabilistic interpretation. A straightforward idea is to identify the probability of conditional statements over a Boolean algebra $\mathbf{B}$ with conditional probability, i.e., with a map $P(\cdot \mid \cdot) : B \times B \setminus \{\bot\} \to [0,1]$ satisfying: (1) $P(b \mid b) = 1$ for all $b \neq \bot$ (normalization); (2) $P(a_1 \vee a_2 \mid b) = P(a_1 \mid b) + P(a_2 \mid b)$ for all a_1, a_2 such that $a_1 \wedge a_2 = \bot$ and $b \neq \bot$; (3) $P(a \mid b) = P(a \wedge b \mid b)$ for all $b \neq \bot$; (4) $P(a \mid c) = P(a \mid b) \cdot P(b \mid c)$ for all $a \leq b \leq c$ and $b, c \neq \bot$ (chain rule), see e.g. [5]. However, Lewis' celebrated triviality result [7] shows that there cannot be a binary operation definable on a Boolean algebra such that it, combined with a probability, would define a conditional probability. More precisely, if conditionals are treated as standard propositions belonging to the same event space as their antecedents and consequents, then the above identification leads only to trivial probabilistic structures.

The approach we analyze here is grounded in the algebraic setting of Boolean algebras, where there is a natural way of formalizing conditional statements starting from the universal algebraic notion of *quotient of an algebra*.

1 Quotients and conditional probabilities

The idea of interpreting conditionals through quotient structures is very natural. Indeed, from an algebraic perspective, quotienting a Boolean algebra $\mathbf{B}$ by the principal filter generated by one of its elements b essentially corresponds to assuming that b is true. In this sense, the quotient algebra $\mathbf{B}/b$ encodes the algebraic counterpart of reasoning under the assumption that b is true. For this reason, related ideas appear in some approaches to conditionals and their probability, like in the case of Calabrese ([1] and [2]) and Mundici ([8]).

The key idea that inspires this approach is that, if we define $P_b : \mathbf{B} \to [0,1]$ as the composition $\pi_b \circ P$, where $\pi_b : \mathbf{B} \to \mathbf{B}/b$ is the natural regular epimorphism of $\mathbf{B}$ to its quotient $\mathbf{B}/b$ and $P : \mathbf{B}/b \to [0,1]$ is any probability function, then we would like to interpret $P(a \mid b)$ as $P_b(a)$. The map P_b satisfies

(1) $P_b(b) = 1$ (normalization);

(2) $P_b(a \vee c) = P_b(a) + P_b(c)$ whenever $a \wedge c = \bot$ (additivity)

(3) $P_b(a) = P_b(a \wedge b)$;

i.e., the first three axioms of conditional probability hold. The only missing axiom for it to be a conditional probability is the chain rule that, conforming with the present notation, reads as follows: for $a \leq b \leq c \in \mathbf{B}$, with $b, c \neq \bot$,

$$P_b(a) \cdot P_c(b) = P_c(a).$$

Notice that this is the only axiom that involves probabilities with different antecedents (i.e., subscripts in our notation). Indeed, the problem with the approach proposed above is that P_b represents the probability of a given a *fixed* antecedent b. In order to define it given *any* antecedent, one should define in this way a family of P_b for any $b \in \mathbf{B}$. However, one can prove the following result which resembles the chain rule.

Proposition 1. *(1) For all $c \in B$, probability $P : \mathbf{B}/c \to [0,1]$ and $b \leq c$, there exists $P^* : \mathbf{B}/b \to [0,1]$ such that for all $a \leq b$, it holds*

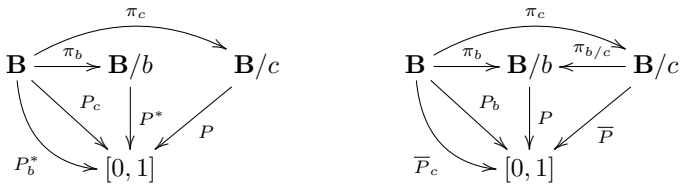
$$P_c(a) = P^*_b(a) \cdot P_c(b)$$

(2) For all $b \in B$, probability $P : \mathbf{B}/b \to [0,1]$ and $c \geq b$, it holds that for all $a \in B$,

$$\overline{P}_c(a) = P_b(a) \cdot \overline{P}_c(b)$$

where $\overline{P} : \mathbf{B}/c \to [0,1]$ is defined as $P \circ \pi_{b/c}$.

The above results can be visualized having a look at the diagrams that are depicted below.



2 Quotient-like conditional algebras

Recently, we investigated a way to internalize the universal algebraic operation of quotient by expanding Boolean algebras with a conditional operator /. Precisely, we define a binary operation / on a Boolean algebra $\mathbf{B}$ such that a/b represents the element a as seen in the quotient $\mathbf{B}/b$ and mapped back to $\mathbf{B}$. The particular structural properties of Boolean algebras allow us to do so in a natural way, indeed, if $b \neq \bot$ the quotient $\mathbf{B}/b$ is a *section* of $\mathbf{B}$, i.e., there is an embedding $\iota_b : \mathbf{B}/b \to \mathbf{B}$ such that $\pi_b \circ \iota_b$ is the identity map.

The idea is then to consider

$$a/b := \iota_b \circ \pi_b(a). \tag{1}$$

We observe that the map ι_b is not uniquely determined, meaning that there can be different embeddings ι, ι' such that $\pi_b \circ \iota = \pi_b \circ \iota'$ is the identity; distinct choices yield distinct values for a/b.

Now, in order to be able to define an operator / over the algebra $\mathbf{B}$, one needs to consider all the different quotients, determined by all choices of elements $b \in B$. Then, if $\bot \neq b \leq c$, by general algebraic arguments one gets a natural way of looking at nested conditionals; indeed it holds that $(\mathbf{B}/c)/\pi_c(b) = \mathbf{B}/b$, which means that $\mathbf{B}/b$ is a quotient of $\mathbf{B}/c$, and actually also

its section. It is then natural to ask that the choices for ι_b and ι_c be *compatible*, in the sense that there is a way of choosing the embedding $\iota_{\pi_c(b)}$ so that

$$\iota_b = \iota_c \circ \iota_{\pi_c(b)}, \tag{2}$$

which yields in particular that the following equation, called *flattening*, holds

$$a/b = (a/b)/c \text{ whenever } b \leq c.$$

The case where $b = \bot$ needs to be considered separately, since the associated quotient is the trivial algebra that cannot be embedded back into $\mathbf{B}$. Since intuitively we are considering the quotients by an element b to mean that "b is true", the *ex falso quodlibet* suggests that we map all elements to $\top$ i.e, $a/\bot := \top$.

We formalized this intuition in the following definitions.

Definition 1. Given a Boolean algebra $\mathbf{B}$, we call a *family of compatible sections on* $\mathbf{B}$ a family of embeddings $\{\iota_b : \mathbf{B}/b \to \mathbf{B} : \bot \neq b \in B\}$, such that:

1. $\pi_b \circ \iota_b = \mathrm{id}_{\mathbf{B}/b}$, and
2. if $\bot \neq b \leq c$ there is an embedding $\iota_{\pi_c(b)} : \mathbf{B}/b \to \mathbf{B}/c$ such that $\pi_{\pi_c(b)} \circ \iota_{\pi_c(b)} = \mathrm{id}_{\mathbf{B}/b}$ and
$$\iota_b = \iota_c \circ \iota_{\pi_c(b)}.$$

Definition 2. A *quotient-like conditional algebra* is an algebra $\mathbf{B} = (B, \wedge, \vee, \neg, /, \bot, \top)$ such that $\mathbf{B}^* = (B, \wedge, \vee, \neg, \bot, \top)$ is a Boolean algebra, and $/$ is a binary operator on $\mathbf{B}$ determined by some family of compatible sections on $\mathbf{B}$, $\{\iota_b : \mathbf{B}^*/b \to \mathbf{B}^* \mid \bot \neq b \in B\}$, by the stipulations:

$$a/b := \iota_b \circ \pi_b(a) \qquad a/\bot := \top,$$

for all $a, b \in B$ with $b \neq \bot$.

These structures then seem to be a suitable algebraic setting in which to frame our initial question involving the probability of quotients, because in this realm we can overcome the limitation pointed out in the previous section on the management of properties of conditional probabilities that involve different antecedents, as it happens for the chain rule.

3 Probabilities of Quotient-like Conditional algebras

In light of the above algebraic analysis, let us now consider a probability function defined on a quotient-like conditional algebra $P : (\mathbf{B}, /) \to [0, 1]$. Since Lewis' triviality result [7] applies to this case as well, and discarding the cases where P is trivial, the combination of P with $/$ results in a map $P(\cdot/\cdot) : B \times B \to [0, 1]$ that in general does not satisfy the axioms of conditional probability. However, the following holds.

Proposition 2. *Every probability function P on an algebra of quotient-like conditionals $(\mathbf{B}, /)$ satisfies the following properties:*

(a) $P(b/b) = \top$ *(normalization)*

(b) $P(a_1 \vee a_2/b) = P(a_1/b) + P(a_2/b)$ *whenever* $a_1 \wedge a_2 = \bot$ *(additivity)*

(c) $P(a/b) = P(a \wedge b/b)$

(d) $P((a/b)/c) = P(a/b)$ *whenever* $b \leq c$ *(flattening)*

In other words, three of the four axioms of conditional probabilities are again satisfied by the combination of P and / and for all probability P. Notice that the last property (d), that is implied by the flattening axiom, can be equivalently written as $P((a/b)/c) = P(a/b \wedge c)$ for $b \leq c$ and hence it corresponds to a weak form of the well-known *import-export property* of conditional probability that is a characteristic property of conditional probabilities.

In light of the above observation we hence put forward the following questions that we aim at investigating in the near future.

(Q1) The probability of a quotient a/b as formalized in a quotient-like conditional algebra does not correspond to updating $P(a)$ via b using Bayesian conditionalization. Indeed, the way $P(a)$ is updated turns out to be a form of *imaging* (see [4]). Imaging can indeed be specified in many different ways and we are interested in understanding what is the effect of the new flattening axiom in this update method.

(Q2) The second line of research we intend to follow aims at providing an axiomatization for the imaging update, in a similar way as the four axioms we recalled at the beginning of this contribution characterize conditionalization. Our conjecture is that imaging on quotients could be captured by the axioms (a)–(d) discussed above.

(Q3) A similar approach to conditionals and conditional probability has been put forward in [3] where it is shown that, roughly speaking, if we avoid embedding the quotients $\mathbf{B}/b$ back to $\mathbf{B}$, one could avoid Lewis's triviality. A deeper analysis between this construction and the one that leads to quotient-like conditionals could eventually clarify at what stage of the algebraic construction Lewis's triviality result originates.

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The effective content of Laver partition theorem

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Laver forcing, and its variants, are an important tool to prove e.g. consistency results concerning cardinal characteristics of the continuum.

Definition 1. A *Laver tree* is a tree $T \subseteq \omega^{<\omega}$ with the property that there exists $\tau \in T$ such that for every $\sigma \in T$ either $\sigma \prec \tau$ or $\tau \preceq \sigma$, and in the latter case σ has infinitely many immediate successors in T. τ is denoted $\mathrm{stem}(T)$.

A *Hechler tree* is a Laver tree T with the additional property that σ has cofinitely many immediate successors in T whenever $\mathrm{stem}(T) \preceq \sigma$.

Laver forcing and *Hechler forcing* are the partial orders of Laver and Hechler trees ordered by inclusion (which implies that $\mathrm{stem}(T) \preceq \mathrm{stem}(S)$ whenever $S \leq T$).

These forcings are used in set theory. One of the key properties of Laver forcing is the following:

Lemma 2 (Pure decision)**.** *Let T be a Laver tree, $k \in \omega$ and let $\{\varphi_i : i < k\}$ be formulas such that $T \Vdash \bigvee_{i<k} \varphi_i$. Then there are $j < k$ and $S \leq T$ with $\mathrm{stem}(T) = \mathrm{stem}(S)$ such that $S \Vdash \varphi_j$.*

This is usually obtained as a consequence of the following fact:

Theorem 3 (Folklore?)**.** *Let $A \subseteq \omega^\omega$ be sufficiently definable and T a Laver tree. There is some $S \leq T$ with $\mathrm{stem}(S) = \mathrm{stem}(T)$ such that either $[S] \subseteq A$ or $[S] \cap A = \emptyset$, where $[S]$ is the set of paths through S.*

We call this theorem Laver partition theorem, or LPT. LPT is actually a special case of the following theorem:

Theorem 4 (Folklore? Arnold Miller?)**.** *Let $A \subseteq \omega^\omega$ be sufficiently definable. Either there is a Laver tree L such that $[L] \subseteq A$ or there is a Hechler tree H such that $[H] \cap A = \emptyset$.*

In an arXiv 2012 preprint ([3]), Miller proves this theorem, which we call LHPT, when $A \in \mathbf{\Sigma}^1_1$.

We are interested in the effective content of LPT and LHPT, and we make two simplifications. Note the following facts:

- if T is a Laver tree with $\mathrm{stem}(T) = \tau$, then $f \mapsto \tau^\frown f$ is a homeomorphism $\omega^\omega \to \tau\omega^\omega$; thus we can identify $[S]$ with $[T]$, where S is "T without the stem";
- any Laver tree T with $\mathrm{stem}(T) = \langle\rangle$ is T-computably isomorphic (as a tree) to the full tree $\omega^{<\omega}$.

So LPT can be stated (up to RCA_0 equivalence) as:

> for every sufficiently definable $A \subseteq \omega^\omega$, there is a Laver tree T such that $\mathrm{stem}(T) = \langle\rangle$ and either $[T] \subseteq A$ or $[T] \cap A = \emptyset$.

and the same reformulation can be done for LHPT.

If we restrict LPT or LHPT to a class of sets $\mathbf{\Gamma}$ then we write $\mathbf{\Gamma}$-LPT and $\mathbf{\Gamma}$-LHPT respectively.

LPT follows quite easily from either the Ramsey property or determinacy, and this holds level by level. In particular

Lemma 5. *In* RCA_0*, either one of Ramsey theorem for open sets and open determinacy implies* $\mathbf{\Sigma}^0_1$-LPT.

The previous lemma does not apply to LHPT (or at least, we do not see how to obtain LHPT from either Ramsey or determinacy). On the other hand, Noam Greenberg formalized Miller's ideas and gave a direct proof of $\mathbf{\Sigma}^0_1$-LPT in ATR_0: this proof actually yields both $\mathbf{\Sigma}^0_1$-LHPT and $\mathbf{\Pi}^0_1$-LHPT.

Recalling that both Ramsey theorem for open sets and open determinacy are equivalent to ATR_0 over RCA_0 ([4, Chapter V]), one can conjecture that $\mathbf{\Sigma}^0_1$-LPT is the "common core" of these two theorems and ask:

Question 6. Is $\mathbf{\Sigma}^0_1$-LPT equivalent to ATR_0, over RCA_0?

The talk, which reports on work in progress, will address this question. Some partial results towards an answer are:

Theorem 7. *Over* RCA_0*, each of* $\mathbf{\Sigma}^0_1$-LHPT*,* $\mathbf{\Pi}^0_1$-LHPT *and* $\mathbf{\Delta}^0_1$-LHPT *is equivalent to* ATR_0*.*

Theorem 8. *Over* $\mathsf{RCA}_0 + \mathbf{\Sigma}^1_1$-$\mathsf{IND}$*,* $\mathbf{\Sigma}^0_1$-LPT *implies* $\mathbf{\Sigma}^1_1$-AC_0*.*

Let HYP be the ω-model consisting of hyperarithmetic sets. Since $\mathsf{HYP} \vDash \mathbf{\Sigma}^0_1$-$\mathsf{AC}_0$ while $\mathsf{HYP} \nvDash \mathsf{ATR}_0$ it is natural to ask:

Question 9. Does $\mathsf{HYP} \vDash \mathbf{\Sigma}^0_1$-$\mathsf{LPT}$?

We can also use Weihrauch reducibility to compare the strength of the partition theorems. We compare the Weihrauch degrees of functions naturally associated to LPT with common benchmarks around the level of ATR_0. First, following the approach of [2, 1] we define the multi-valued functions associated to LPT.

Definition 10. For any $\Gamma \in \{\mathbf{\Sigma}^0_1, \mathbf{\Pi}^0_1, \mathbf{\Delta}^0_1\}$, we let

- $\mathsf{wFindL}_\Gamma :\subseteq \Gamma(\omega^\omega) \rightrightarrows \omega^\omega$ mapping A to $\{T : T \text{ is a Laver tree and } [T] \subseteq A\}$, with $\mathrm{dom}(\mathsf{wFindL}_\Gamma) = \{A \in \Gamma(\omega^\omega) : \forall T\ (T \text{ is a Laver tree} \to \exists g \in [T]\ g \notin A)\}$;
- $\mathsf{FindL}_\Gamma :\subseteq \Gamma(\omega^\omega) \rightrightarrows \omega^\omega$ mapping A to $\{T : T \text{ is a Laver tree and } [T] \subseteq A\}$, with $\mathrm{dom}(\mathsf{FindL}_\Gamma) = \{A \in \Gamma(\omega^\omega) : \exists T\ (T \text{ is a Laver tree} \wedge [T] \subseteq A)\}$;
- $\mathsf{LL}_\Gamma \colon \Gamma(\omega^\omega) \rightrightarrows \omega^\omega$ mapping A to $\{T : T \text{ is a Laver tree and } [T] \subseteq A \vee [T] \cap A = \emptyset\}$.

Theorem 11.

- $\mathsf{UC}_{\omega^\omega} \equiv_{\mathrm{sW}} \mathsf{wFindL}_{\mathbf{\Delta}^0_1} \equiv_{\mathrm{sW}} \mathsf{wFindL}_{\mathbf{\Sigma}^0_1} \equiv_{\mathrm{sW}} \mathsf{LL}_{\mathbf{\Delta}^0_1} \equiv_{\mathrm{sW}} \mathsf{FindL}_{\mathbf{\Delta}^0_1} \equiv_{\mathrm{sW}} \mathsf{FindL}_{\mathbf{\Sigma}^0_1}$*;*
- $\mathsf{FindL}_{\mathbf{\Pi}^0_1} \equiv_{\mathrm{sW}} \mathsf{C}_{\omega^\omega}$*;*
- $\mathsf{C}_{2^\omega} \star \mathsf{wFindL}_{\mathbf{\Pi}^0_1} \equiv_{\mathrm{W}} \mathsf{C}_{\omega^\omega}$ *and hence* $\mathsf{wFindL}_{\mathbf{\Pi}^0_1} \equiv^{\mathrm{a}}_{\mathrm{W}} \mathsf{C}_{\omega^\omega}$*.*

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A topos for extended Weihrauch degrees

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Categorical methods are applied in many areas of mathematical logic, however an area that has been less influenced by this categorical approach is computability, although there is a long tradition of application of categorical methods in realizability studies, see e.g. [11].

In recent years, there have been some works starting to approach computability-like notions from a categorical perspective (see e.g. [6, 5, 9, 10]). In [1], Bauer introduced an abstract notion of reducibility between predicates, called instance reducibility, which commonly appears in reverse constructive mathematics. In a relative realizability topos, the instance degrees correspond to a generalization of (realizer-based) Weihrauch reducibility, called *extended Weihrauch degrees*, and the "classical" Weihrauch degrees [3] correspond precisely to the $\neg\neg$-dense modest instance degrees in Kleene-Vesley realizability. Upon closer inspection, it is not hard to check that realizer-based Weihrauch reducibility is a particular case of Bauer's notion.

The main goal of this talk, which is based on the recently published paper [7], is to show how one can define a topos for extended Weihrauch degrees, providing a suitable universe for studying this reducibility categorically. Then, we take advantage from this categorical presentation, and we establish the precise connection between extended Weihrauch degrees and realizability.

The main tools we adopt to construct such a topos are: (i) the *tripos-to-topos construction* [4], which produces a topos from a given tripos (that is, a particular kind of Lawvere hyperdoctrine which has enough structure to deal with higher-order logic properly), and (ii) the *(full) existential completion*, a construction that freely adds left adjoints along all the morphisms of the base of a given doctrine.

In fact, we first define a doctrine $\mathfrak{iR}$ over the category of partitioned assemblies abstracting Bauer's notion of instance reducibility between realizability predicates, and we prove that it is a tripos. This doctrine is defined as the (full) existential completion $\mathfrak{eiR}^{\exists}$ of a more basic doctrine $\mathfrak{eiR}$, following the same idea used in [10] for defining doctrines abstracting computability reducibility.

Then we introduce a second doctrine $\mathfrak{eW}$ which provides a direct categorification of the notion of extended Weihrauch degrees and we prove that $\mathfrak{eW} \cong \mathfrak{iR}$. This equivalence shows in particular that $\mathfrak{eW}$ is a tripos.

This result can be seen as a fibrational version of Bauer's result showing that Weihrauch reductions and instance reductions are equivalent [1]. Indeed, the order of the fibres of $\mathfrak{iR}$ corresponds to the notion of instance reduction, while that of $\mathfrak{eW}$ corresponds to the extended Weihrauch reduction.

Finally, once we have proved that the extended Weihrauch doctrine is a tripos, we can define the topos of extended Weihrauch degrees as the topos obtained by applying the tripos-to-topos to the tripos $\mathfrak{eW}$ and study the connections with (relative) realizability toposes. In particular, we prove that the relative realizability topos $\mathsf{RT}[\mathbb{A}, \mathbb{A}']$ [2] is equivalent to a topos $\mathsf{sh}_j(\mathsf{EW}[\mathbb{A}, \mathbb{A}'])$ of j-sheaves for a Lawvere-Tierney topology j over $\mathsf{EW}[\mathbb{A}, \mathbb{A}']$. To prove this result, there are two facts playing a key role: first that the extended Weihrauch tripos is a (full) existential completion; second, that realizability toposes can be presented as exact completions of the category of partitioned assemblies [8].

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Exponential Rings and coextensivity

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1 Introduction

The good properties of the geometry of definable sets in exponential structures suggest that one should be able to extend some techniques of algebraic geometry to exponential algebra. Attempts have been made in e.g. [6, 2]. A categorical route to import some of Grothendieck's algebraic-geometric technology, built by Lawvere upon [3], singles out *coextensivity* of the ambient category of algebras as a first structural prerequisite: it guaranties that finite sums in the dual category of "spaces" behave as disjoint unions and are stable under pullback.

In this extended abstract, we (i) show that the category of exponential rings is not coextensive, and (ii) isolate natural subcategories that are coextensive. These subcategories are *natural* in the sense that they are motivated by requirements like "having enough clopens", "having enough points" or suggested by the relationship with C^∞-rings.

2 E-rings and their decompositions

Let $\mathcal{A}$ be a category of algebras (rings, Boolean algebras, etc.). One may wonder when its opposite $\mathcal{A}^{\mathrm{op}}$ can be regarded as a category of "spaces". A basic observation is that the finite coproducts in $\mathcal{A}^{\mathrm{op}}$ should represent finite disjoint unions of spaces. In [5] Lawvere propose the axiom of extensivity to formalize the fact that "unions of spaces are well behaved". In an extensive category (see [1]) coproducts are disjoint (inclusions are monomorphisms with empty intersection) and are stable under pullback. E.g. the $\mathbf{Ring}^{\mathrm{op}}$ is extensive. Lawvere stressed that extensivity is a key first step in Grothendieck's passage from affine schemes to "gros" toposes. This perspective suggests the natural question: *Is the category* E**Ring** *of exponential rings coextensive?*

Definition 1 (See e.g., [9])**.** An *exponential ring* is a commutative unitary ring A equipped with a function $\mathrm{E}\colon A \to A$ satisfying

$$\mathrm{E}(0) = 1, \qquad \mathrm{E}(x+y) = \mathrm{E}(x)\mathrm{E}(y) \quad \text{for all } x, y \in A,$$

so that $\mathrm{E} : (A, +) \to (A^\times, \cdot)$ is a group homomorphism into the multiplicative group of units.

Most of our results hold in the more general category E**Rig** of *exponential rigs*, defined analogously over rigs (recall that a *rig* is a commutative semiring with unit).

A first step toward coextensivity is to understand the products in E**Rig** and how they interact with the pushouts. As in the classical case of rings and rigs, finite product decompositions are determined by complemented elements.

In a rig A, an element $a \in A$ is *complemented* if there exists $a' \in A$ such that

$$aa' = 0, \qquad a + a' = 1.$$

*Presenting author

In a ring, complemented elements coincide with idempotents, with $a' = 1 - a$. In E**Rig** there is an additional compatibility condition with E.

Let A be an E-rig and let (a) be the principal ideal generated by a in the underlying rig. We say that (a) is an E-*ideal* if the quotient $A/(a)$ carries a unique E-structure for which the quotient map $A \to A/(a)$ is a morphism of E-rigs. A convenient characterization is the following: if a is complemented, with complement a', then (a) is an E-ideal iff

$$a' \,\mathrm{E}(ax) = a' \quad \text{for all } x \in A,$$

equivalently iff

$$\mathrm{E}(ax) + a = 1 + a\,\mathrm{E}(ax) \quad \text{for all } x \in A.$$

We call such an a *weakly* E-*complemented*, and call a E-*complemented* if both a and a' are weakly E-complemented.

The basic structural fact is that binary products in E**Rig** are classified by E-complemented elements exactly as products in **Rig** are classified by complemented elements.

Lemma 2. *A span $A \to D$, $A \to E$ is a product in* E**Rig** *iff it is (up to isomorphisms) the canonical span*

$$A \longrightarrow A/(a), \qquad A \longrightarrow A/(a')$$

with $a \in A$ E-complemented.

3 Failure of coextensivity

Call $z \in A$ a *formal zero* if $\mathrm{E}z = 1$.

Definition 3. An E-rig is *extreme* if all its elements are formal zeros, i.e. if E is constantly 1.

Extreme E-rigs form a full subcategory of E**Rig** canonically isomorphic to **Rig** (send a rig A to the same underlying rig with $\mathrm{E} \equiv 1$). In extreme E-rigs, every complemented element is automatically E-complemented.

Let A denote the E-ring $\mathbb{Z}^3$ endowed with the function $\mathrm{E}(m,n,p) = (1,1,(-1)^{m-n})$. The idempotent element $a = (1,1,0) \in A$ is a formal zero ($\mathrm{E}a = 1$), but it is not weakly E-complemented: taking $x = (1,0,0)$ one checks that

$$\mathrm{E}(ax) + a \neq 1 + a\,\mathrm{E}(ax).$$

Thus, a is complemented but not E-complemented.

Let $\mathbb{Z}_e$ denote $\mathbb{Z}$ equipped with the extreme E-structure ($\mathrm{E} \equiv 1$). In E**Ring**, the product $\mathbb{Z}_e \times \mathbb{Z}_e$ has the usual idempotents $(1,0)$ and $(0,1)$, and these are E-complemented since the object is extreme.

The idempotent $a \in A$ determines a unique morphism of E-rings

$$f \colon \mathbb{Z}_e \times \mathbb{Z}_e \longrightarrow A$$

sending $(1,0)$ to a (and hence $(0,1)$ to $1-a$). One can compute the E-ideals generated by a and by $1-a$ in A and show that the pushouts of the projections

$$\mathbb{Z}_e \longleftarrow \mathbb{Z}_e \times \mathbb{Z}_e \longrightarrow \mathbb{Z}_e$$

along f yield a span

$$\mathbb{Z}_2 \longleftarrow A \longrightarrow \mathbb{Z} \times \mathbb{Z}$$

(where the codomains carry their induced extreme E-structures). Since the class of extreme E-rings is closed under finite products, if this pushout span were a product then A would have to be extreme. But it is not. Therefore, products are not stable under pushout, and we obtain:

Theorem 4. *The categories* E**Rig** *and* E**Ring** *are not coextensive.*

4 Coextensive subcategories

The negative result motivates the search for natural coextensive subcategories. A geometric intuition motivating the failure of extensivity is that the dual space of an E-ring may have fewer clopen decompositions than the underlying affine scheme of the ring obtained by forgetting E. One basic idea is to isolate those E-objects for which the clopen decompositions correspond to the ones obtained by forgetting E.

Definition 5. Let $F\colon \mathcal{C} \to \mathcal{D}$ be a functor between categories with finite coproducts. An object X of $\mathcal{C}$ is *F-crisp* if F *creates summands of* FX: for every summand $V \to FX$ there exists a unique summand $U \to X$ mapped to it by F. Write $\mathrm{Cr}(F, \mathcal{C})$ for the full subcategory of F-crisp objects.

Taking $G\colon$ E**Rig** $\to$ **Rig** the forgetful functor and dualizing, one finds that an E-rig is co-crisp (in E**Rig**) precisely when every complemented element is E-complemented, i.e. when the quasi-equational implication

$$(u + v = 1) \wedge (uv = 0) \;\vdash\; u\,\mathrm{E}(vx) = u$$

holds. Similarly for E-rings.

We give a general sufficient criterion ensuring that $\mathrm{Cr}(F, \mathcal{C})$ is extensive, under appropriate conditions. Applying this criterion to the dual of the forgetful functor E**Rig** $\to$ **Rig** yields the following:

Theorem 6. *The quasi-varieties of co-crisp* E*-rigs and co-crisp* E*-rings are coextensive. Moreover, the forgetful functors to* **Rig** *and* **Ring** *preserve and reflect finite products on these subcategories.*

A second geometric intuition is that definable spaces often have enough points in the sense that evaluation at points is jointly epic. So, one is led to consider those objects determined by their points.

In a category with an initial object 0, call an object A *point-determined* if the family of maps $A \to 0$ is jointly monic. In coslice categories such as $\mathbb{R}/$E**Ring**, where objects are E-homomorphisms $\mathbb{R} \to A$, the initial object is the identity $\mathbb{R} \to \mathbb{R}$, so these maps correspond to retractions $A \to \mathbb{R}$.

We show that point-determined objects form a coextensive subcategory. We prove that in any extensive category with connected terminal object, the full subcategory generated under quotients by copowers of 1 is extensive. Dualizing this yields:

Theorem 7. *The categories of point-determined objects in* E**Ring** *and in* $\mathbb{R}/$E**Ring** *are coextensive.*

Finally, we connect exponential rings to C^∞-rings (see [4, 7]) and import nilpotent information. For an E-rig K, the rig of dual numbers $K[\epsilon]$ (see [8]) carries a canonical E-structure defined by

$$\mathrm{E}(a + b\epsilon) = \mathrm{E}(a) + \mathrm{E}(a)\,b\epsilon,$$

which is suggested by the standard properties $\exp(a+b\epsilon) = \exp(a)\exp(b\epsilon)$ and $\exp(b\epsilon) = 1+b\epsilon$.

On the smooth side, every Weil algebra W carries a canonical C^∞-ring structure, and any $\mathbb{R}$-algebra morphism $W \to A$ into a C^∞-ring is automatically C^∞. Forgetting along C^∞-**Ring** $\to \mathbb{R}/\mathrm{E}$**Ring** yields:

Lemma 8. *Every Weil algebra has a canonical structure of real exponential ring, functorial with respect to $\mathbb{R}$-algebra maps into C^∞-rings.*

Motivated by this, we call an object of $\mathbb{R}/\mathrm{E}$**Ring** *Weil-determined* if it is a subobject of a product of Weil algebras. Since Weil algebras are directly indecomposable (hence co-crisp), the general "enough quotients" argument inside the co-crisp subcategory implies:

Theorem 9. *The category of Weil-determined real exponential rings is coextensive.*

The failure of coextensivity for E**Ring** indicates that the geometry of exponential algebra cannot be developed naively. The coextensive subcategories identified here are natural restrictions to further develop the geometry of exponential algebra in the style of Grothendieck.

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A game characterization for weak continuous reducibility

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In this talk, I present a game that characterizes the weak continuous reducibility of functions between recursively presented metric spaces to functions between the Baire space ω^ω. This work uses the structure of continuous degrees in Effective Descriptive Set Theory to generalize part of the results presented in [Lut23] to functions between recursively presented metric spaces.

Definition 1. Given X_f, Y_f, X_g, Y_g topological spaces and two functions $f : X_f \to Y_f$ and $g : X_g \to Y_g$, we say that f is *weakly continuous reducible* to g ($f \leqslant_w g$) if there exist a continuous function $\varphi : X_f \to X_g$ and a partial continuous function $\psi : Y_g \times X_f \rightharpoonup Y_f$[1] such that $\forall x \in X_f(f(x) = \psi(g(\varphi(x)), x))$.

Weak reducibility is also considered in Computable Analysis, where it is called *continuous Weihrauch reducibility*.

Definition 2. Let (X, d) be a separable metric space and $\boldsymbol{r} = (r_i)_{i\in\omega}$ an enumeration (possibly with repetitions) of a dense subset of X. We say that $\boldsymbol{r}$ is a *recursive presentation* of X if the relations on ω^3

$$P(i,j,k) \Leftrightarrow \mathrm{d}(r_i, r_j) \leqslant q_k$$
$$Q(i,j,k) \Leftrightarrow \mathrm{d}(r_i, r_j) < q_k$$

are computable. The structure $(X, \mathrm{d}, \boldsymbol{r})$ is called *recursively presented metric space*. If moreover (X, d) is complete, then $(X, \mathrm{d}, \boldsymbol{r})$ is called *recursively presented Polish space*.

Given two functions $f : \omega^\omega \to \omega^\omega$ and $g : \mathcal{X}_g \to \mathcal{Y}_g$ where $\mathcal{X}_g$ and $\mathcal{Y}_g$ are recursively presented metric spaces, we define a two-player, zero-sum, perfect information game $G_M(f, g)$. This game characterizes weak reducibility between f and g in the sense that Player 2 has a winning strategy if and only if $f \leqslant_w g$. To get such a result, we modify the original definition of Lutz's game using the admissible representation $\rho_{\mathcal{X}} : \omega^\omega \rightharpoonup \mathcal{X}$ for recursively presented metric spaces $\mathcal{X} = (X, \mathrm{d}, \boldsymbol{r})$ defined as:

$$\rho_{\mathcal{X}}(p) = x \Leftrightarrow \mathrm{ran}(p) = \mathrm{N}_{\mathrm{base}}(x) \coloneqq \{n \in \omega \mid x \in V_n^{\mathcal{X}}\}$$

where $V_i^{\mathcal{X}}$ is the i-th ball associated to the recursive presentation of $\mathcal{X}$ (i.e.,$V_i^{\mathcal{X}} = B_{q_{(i)_1}}(\boldsymbol{r}_{(i)_0}) = \{x \in X \mid d(x, \boldsymbol{r}_{(i)_0}) < q_{(i)_1}\}$[2]), and $(q_l)_{l\in\omega}$ is a fixed recursive enumeration of $\mathbb{Q}^+$. That is, we identify each point $x \in X$ with the elements of ω^ω that correspond to an enumeration of the neighborhood basis given by the balls with rational radius and centers in the dense subset $\boldsymbol{r}$. We use such representation because it has nice (effective) topological properties, namely: it is Σ^0_1-recursive on its domain, open, surjective and with Polish fibers (see [Peq15, Theorem 1]) and, moreover, satisfies the following property [GKN21, Fact 2.4 (iv)] w.r.t. the structure of continuous degrees:

$$\forall x \in X \forall z \in \omega^\omega (x \leqslant_M z \Leftrightarrow \exists p \in \omega^\omega (\rho_{\mathcal{X}}(p) = x \wedge p \leqslant_T z))$$

Joint work with Raphaël Carroy

[1] We denote with the arrow $\rightharpoonup$ all partial functions.

[2] We denote with $()_0, ()_1 : \omega \to \omega$ the two functions which associate to the code of a pair the first and the second element (respectively).

Where $\leqslant_M$ denotes the representation reducibility defined as in [GKN21, Definition 2.3] and [Mil04, Definition 3.1]. This is a generalization of Turing reducibility to the context of recursively presented metric spaces.
Given a topological pointclass $\mathbf{\Gamma}$, for any function $f : X_f \to Y_f$ between topological spaces we write $f \in \mathrm{dec}(\mathbf{\Gamma})$ if there exists a partition $\{A_m\}_{m\in\omega}$ of X such that $f\restriction A_m$ is $\mathbf{\Gamma}$-measurable for all $m \in \omega$. In particular, $f \in \mathrm{dec}(\mathbf{\Sigma}^0_1)$ coincides with f being σ-continuous.

Recall that the *Turing Jump* $J : \omega^\omega \to 2^\omega$ is the function which associates to each $x \in \omega^\omega$ the characteristic function corresponding to the x-relativized diagonalized Halting Problem $x' = \{e \in \omega \mid \varphi^x_e(e)\downarrow\}$. This function is $\mathbf{\Sigma}^0_2$-measurable, injective, and not σ-continuous. It can be iterated by composition, and the n-th iteration of the Turing Jump $J^{(n)} : \omega^\omega \to 2^\omega$ is $\mathbf{\Sigma}^0_{n+1}$-measurable but not in $\mathrm{dec}(\mathbf{\Sigma}^0_n)$.
Using the game G_M and the iterated Turing Jump, we extend the following dichotomy, which was originally proved in [Lut23] for functions on the Baire space:

Theorem 3. Given $\mathcal{X}$ and $\mathcal{Y}$ recursively presented metric spaces, and $g : X \to Y$ function such that $G_M(J^{(n)}, g)$ is determined, then either $g \in \mathrm{dec}(\mathbf{\Sigma}^0_n)$ or $J^{(n)} \leqslant_w g$.

For the case $n = 1$, this dichotomy was originally proved by Solecki in [Sol98] for Baire class 1 functions, although there it is formulated using the *Pawlikowski function* (which is $\equiv_w$-equivalent to the Turing Jump) and *topological embeddability*, which is a stronger form of reducibility between functions. Then, it was extended to all functions with analytic graphs by Pawlikowski and Sabok (see [PS12]).

In addition, we prove that $G_M(J^{(n)}, g)$ is always determined when g is a Borel function and is defined on a recursively presented Polish space.

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On Ditor's Problem

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Given a positive integer n, an n-ladder is a lower finite lattice whose elements have at most n lower covers. The notion of an n-ladder was introduced independently by Ditor [1] and Dobbertin [2] under different names[1]. Ditor showed that every n-ladder has cardinality at most $\aleph_{n-1}$ [1], and asked whether this cardinality bound is sharp (see also [3, p. 291]):

Ditor's Problem. *For every $n > 0$, is there an n-ladder of cardinality $\aleph_{n-1}$?*

The case $n = 1$ is trivial: ω with its usual ordering is a 1-ladder of cardinality $\aleph_0$. Ditor proved that the answer to his question is also positive when $n = 2$, i.e., there exists a 2-ladder of cardinality $\aleph_1$. Since then, 2-ladders have been used primarily, but not exclusively, in representation problems in universal algebra for structures of cardinality $\leq \aleph_1$ (e.g. [2, 4, 9, 7]).

Recent progress has been made on the remaining cases of Ditor's problem. Wehrung [8] proved that the existence of a 3-ladder of cardinality $\aleph_2$ follows from either of two independent set-theoretic assumptions: $\mathsf{MA}_{\omega_1}(\aleph_1\text{-precaliber})$, that is, MA_{ω_1} restricted to forcings of precaliber $\aleph_1$; and the existence of an $(\omega_1, 1)$-morass.

We proved in [5] that for every $n > 2$ the existence of an n-ladder of cardinality $\aleph_{n-1}$ follows from $\square_{\omega_1} + \square_{\omega_2} + \ldots + \square_{\omega_{n-2}}$, i.e., from Jensen's $\square_\kappa$ holding at the first $n-2$ uncountable cardinals. In particular, since the axiom of constructibility $\mathsf{V} = \mathsf{L}$ implies $\square_\kappa$ for every uncountable cardinal κ, Ditor's problem has a positive solution for every n under $\mathsf{V} = \mathsf{L}$.

In this talk we survey these results and discuss the recently introduced notion of maximal n-ladder [6]: an n-ladder is maximal if it is not isomorphic to a proper ideal of an n-ladder. This notion stems naturally from the works of Ditor and Wehrung [1, 8], and plays a central role in Ditor's problem and related questions. We then present recent results from [6] on the existence and forcing-theoretic properties of maximal n-ladders.

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On the Wadge hierarchy of Borel subspaces of the Baire space

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Given a topological space X and $A, B \subseteq X$, one says that A is *Wadge reducible* to B and writes $A \leq_W^X B$ if there is a continuous function $f\colon X \to X$ such that $f^{-1}(B) = A$. The relation $\leq_W^X$ is a quasi-order on $\mathscr{P}(X)$, inducing the equivalence relation $\equiv_W^X$. The *Wadge degree* of a subset $A \subseteq X$ is its $\equiv_W^X$-equivalence class, while the *Wadge class* generated by A is $A{\downarrow_X} := \{B \subseteq X \mid B \leq_W^X A\}$, which is a boldface pointclass in X (i.e., a subset of $\mathscr{P}(X)$ closed under continuous preimages). The *Wadge hierarchy* $\mathcal{W}_X$ of the space X is the quotient set $\mathscr{P}(X)/\equiv_W^X$, with the order relation induced by $\leq_W^X$ (still denoted with the same symbol). The Wadge hierarchy of X is a complexity hierarchy on the space X: when considering the Borel subsets of X, for example, the Wadge hierarchy is a refinement of the Borel one.

In his doctoral thesis (3), William Wadge studied what is now called the Wadge hierarchy of the Baire space ω^ω thanks to the *Wadge game*, a game for two players characterizing Wadge reducibility in ω^ω. He proved the well-known Wadge's Lemma: assuming the Axiom of Determinacy (AD), for any $A, B \subseteq \omega^\omega$, it either holds $A \leq_W^{\omega^\omega} B$ or $\omega^\omega \setminus B \leq_W^{\omega^\omega} A$. As a consequence, antichains with respect to the Wadge order have size at most two; hence, in the Wadge hierarchy, we only find selfdual degrees (the degree of some $A \subseteq \omega^\omega$ is *selfdual* if $A \leq_W^{\omega^\omega} \omega^\omega \setminus A$) or nonselfdual pairs of degrees. Moreover, the relation $\leq_W^{\omega^\omega}$ was proved by Martin and Monk to be well-founded, so that we can talk about the *Wadge rank* of a Wadge degree. To summarize these two results, one says that $\mathcal{W}_{\omega^\omega}$ is a *semi-well-order*. So, describing $\mathcal{W}_{\omega^\omega}$ consists in finding the following two things: its length and the positions where we find a nonselfdual pair of degrees (in opposition to a selfdual degree). Under AD, a complete description of the Wadge hierarchy of ω^ω was achieved: the minimal Wadge degrees are $\{\emptyset\}$ and $\{\omega^\omega\}$, constituting a nonselfdual pair; then, nonselfdual pairs and selfdual degrees alternate – we say that the *alternating property* holds. Moreover, if the Wadge rank of some $A \subseteq \omega^\omega$ is a limit ordinal with countable (respectively, uncountable) cofinality, then A is selfdual (respectively, nonselfdual). Finally, the length of $\mathcal{W}_{\omega^\omega}$ was proved to be $\Theta := \sup\{\alpha \mid \exists f(f\colon \omega^\omega \twoheadrightarrow \alpha)\}$.

The recent work (1) by Carroy, Motto Ros, and Scamperti gives, under AD, a complete classification of $\mathcal{W}_X$ for an arbitrary zero-dimensional Polish X. For such an X, $\mathcal{W}_X$ is still a semi-well-order, and the alternating property holds. Moreover, while the Wadge hierarchy collapses before the class $\mathbf{\Delta}_2^0(X)$ whenever X is countable, the length of $\mathcal{W}_X$ coincides with the length of $\mathcal{W}_{\omega^\omega}$ for any uncountable zero-dimensional Polish X. In this last case, the (non)selfduality of a subset $A \subset X$ with limit Wadge rank depends on the cofinality of such a rank and also on the particular space X.

As a zero-dimensional Polish space is homeomorphic to a closed subspace of ω^ω, one may wonder what happens when we consider a more complex subspace, such as an arbitrary Borel set. In this talk, working in ZF+DC+AD, we will present the first steps towards this investigation. We will start from the simplest possible case, the case of a countable perfect $X \in \mathbf{\Sigma}_2^0(\omega^\omega) \setminus \mathbf{\Pi}_2^0(\omega^\omega)$. Since any such space is homeomorphic to $\mathbb{Q}$, it is enough to focus on the rationals. Even though $\mathbb{Q}$ is a countable space, we see that the hierarchy of differences of open sets does not collapse in it, so that in particular $\mathcal{W}_{\mathbb{Q}}$ has uncountable length. This is a first difference with the Polish case: indeed, in (1)

it is shown that $\mathcal{W}_X$ has countable length for every countable zero-dimensional Polish space X. Moreover, unlike in all zero-dimensional Polish spaces with a non-collapsing difference hierarchy (for example, the Baire space), the following holds in $\mathbb{Q}$.

Proposition. *There is $A \subseteq \mathbb{Q}$ such that $\Delta(\mathbf{\Sigma}^0_2(\mathbb{Q})) = A{\downarrow_{\mathbb{Q}}}$. Therefore, the class $\Delta(\mathbf{\Sigma}^0_2(\mathbb{Q}))$ is a Wadge class.*

Hence, the space $\mathbb{Q}$ witnesses that there is at least a Borel subspace X of ω^ω in which there is a *new* Wadge class, i.e., a boldface class not admitting a complete set when considered in ω^ω (or in any zero-dimensional Polish space), but having one when considered in X.

Afterwards, we will present some partial results about the Wadge hierarchy of an arbitrary $X \in \mathbf{Bor}(\omega^\omega) \setminus \mathbf{\Pi}^0_2(\omega^\omega)$. Thanks to the Hurewicz Theorem, guaranteeing the existence of a closed copy of $\mathbb{Q}$ in X, we manage to transfer to X all the results obtained for the rationals. Therefore, we get a complete description of the Wadge hierarchy of X below $\Delta(\mathbf{\Sigma}^0_2(X))$ – and then of the whole Wadge hierarchy, if X is countable. In particular:

Theorem. *Consider $X \in \mathbf{Bor}(\omega^\omega) \setminus \mathbf{\Pi}^0_2(\omega^\omega)$. The class $\Delta(\mathbf{\Sigma}^0_2(X))$ is a Wadge class.*

When restricting to an uncountable $X \in \mathbf{Bor}(\omega^\omega) \setminus \mathbf{\Pi}^0_2(\omega^\omega)$, we see that we can determine the nonselfdual part of the Wadge hierarchy by generalizing the analogous result for uncountable zero-dimensional Polish spaces presented in (1). We denote with $\mathrm{Class_W}(X)$ (respectively, $\mathrm{Class_{NSD}}(X)$ and $\mathrm{Class_{SD}}(X)$) the collection of all (respectively, all nonselfdual and all selfdual) Wadge classes in X.

Theorem. *Let $X \subseteq \omega^\omega$ be uncountable Borel. Then the map*

$$\Phi\colon \langle \mathrm{Class_{NSD}}(\omega^\omega), \subseteq\rangle \to \langle \mathrm{Class_{NSD}}(X), \subseteq\rangle, \mathbf{\Gamma} \mapsto \mathbf{\Gamma}{\restriction}X = \{A \cap X \mid A \in \mathbf{\Gamma}\}$$

is an isomorphism, and in particular $\mathrm{Class_{NSD}}(X) = \{\mathbf{\Gamma}{\restriction}X \mid \mathbf{\Gamma} \in \mathrm{Class_{NSD}}(\omega^\omega)\}$.

According to this theorem, the only differences between the Wadge hierarchies of ω^ω and of an uncountable Borel $X \subseteq \omega^\omega$ involve selfdual Wadge classes. In particular, since under certain assumptions on X we see that $\langle \mathrm{Class_{SD}}(\omega^\omega), \subseteq\rangle$ embeds into $\langle \mathrm{Class_{SD}}(X), \subseteq\rangle$, determining the new selfdual Wadge classes would give a complete description of the Wadge hierarchy of X.

Finally, we will describe the Wadge hierarchy of a specific uncountable $X \in \mathbf{Bor}(\omega^\omega) \setminus \mathbf{\Pi}^0_2(\omega^\omega)$. In this example, we can see that whether the class $\Delta(\mathbf{\Gamma}(X))$ is a new Wadge class or not depends on two invariants associated (by Louveau, in (2)) to a class $\mathbf{\Gamma} \in \mathrm{Class_{NSD}}(\omega^\omega)$, the *level* and the *type*.

This is joint work with Riccardo Camerlo and Raphaël Carroy.

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On the Rudin-Frolík ordering of ultrafilters on Boolean algebras

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Abstract

The Rudin-Frolík ordering of ultrafilters over ω was introduced by Frolík [2], who used it to show that the topological space $\beta\omega \setminus \omega$ is not homogeneous. Two generalizations of the Rudin-Frolík ordering to ultrafilters on complete Boolean algebras have been proposed independently by Balcar and Dow [1] and Murakami [3]. In this talk, we compare these two notions and investigate to what extent they coincide in general. This is a joint work in progress with Jörg Brendle (Kobe University).

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Automorphism groups of structures with the Lascar Property

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This is joint work with Gianluca Paolini.

1 Reconstruction Theory

Model theory lies at the intersection of logic and algebra, and studies mathematical structures through the lens of first-order theories and their models. A central theme in the subject is to understand how much information about a structure is encoded in its automorphism group. This perspective gives rise to *reconstruction theory*, which investigates to what extent a structure can be recovered from its automorphism group, and conversely how properties of the structure are reflected in that group. This area connects model theory with group theory and descriptive set theory, and originates in the seminal work of Ahlbrandt and Ziegler [AZ86].

Theorem. *Let M and N be ω-categorical structures. Then*

$$\mathrm{Aut}(M) \cong_{\mathrm{top}} \mathrm{Aut}(N) \iff M \text{ and } N \text{ are bi-interpretable.}$$

In [Pao24], motivated by a question from descriptive set theory posed in [NST21], the author proved that, under the assumption of weak elimination of imaginaries, the open subgroups of a countable ω-categorical structure M admit a combinatorial description as "generalized setwise stabilizers" of finitely generated algebraically closed subsets of M. This result led both to partial progress on the problem from [NST21], and to a concrete description of the group $\mathrm{Aut}_{\mathrm{top}}(\mathrm{Aut}(M))$ of topological automorphisms of $\mathrm{Aut}(M)$, where $\mathrm{Aut}(M)$ is equipped with the topology of pointwise convergence inherited from $\mathrm{Sym}(\omega)$.
The goal of our paper [PP26] is to extend the techniques developed in [Pao24] beyond the ω-categorical setting, establishing connections with classical work of Evans and Lascar on outer (topological) automorphisms of automorphism groups of countable saturated models of strongly minimal theories.

1.1 Results

We weaken the main hypothesis of [Pao24] as follows.

Definition 1.1. *We say that a structure M has the* Lascar property *if for every open subgroup $H \leq \mathrm{Aut}(M)$, there exists K finitely generated algebraically closed subset of M such that*

$$\mathrm{Aut}(M)_{(K)} \leq H \leq \mathrm{Aut}(M)_{\{K\}}.$$

Theorem 1.2. *Let M and N be countable saturated structures with the Lascar Property. Then*

1. *pointwise stabilizers of finitely generated algebraically closed subsets of M are G_δ-subgroups of $\mathrm{Aut}(M)$, and there is a Galois correspondence between them and the set of finitely generated algebraically closed subsets of M;*

2. *the structure* $\mathcal{E}_M^{\mathrm{ex}}$ *associated to* M *is such that:*

$$\mathrm{Aut}(M) \cong_{\mathrm{top}} \mathrm{Aut}(N) \Leftrightarrow \mathcal{E}_M^{\mathrm{ex}} \cong \mathcal{E}_N^{\mathrm{ex}};$$

and

$$\mathrm{Aut}_{\mathrm{top}}(\mathrm{Aut}(M)) \cong \mathrm{Aut}(\mathcal{E}_M^{\mathrm{ex}}).$$

The structure $\mathcal{E}_M^{\mathrm{ex}}$ consists of finitely generated algebraically closed subsets of M, together with partial isomorphisms between them, and encodes the combinatorial properties of the automorphism group $\mathrm{Aut}(M)$.
All saturated models of strongly minimal theories satisfy the Lascar property. In this framework, we recover a result of Lascar and Evans on algebraically closed fields of characteristic zero [EL97].

Corollary 1.3. *Let* $M \models \mathrm{ACF}_0$ *be countable and saturated. Then*

$$\mathrm{Aut}(\mathrm{Aut}(M)) \cong \mathrm{Aut}(\mathbb{G}_M),$$

and every automorphism of $\mathrm{Aut}(M)$ *is inner. In particular,* $\mathrm{Aut}(M)$ *is complete.*

The next corollary appears to be new.

Corollary 1.4. *Let* V *be a vector space of dimension* $\aleph_0$ *over a countable field* $\mathbb{K}$*. Then:*

$$\mathrm{Aut}(\mathrm{GL}(V)) \cong \mathrm{Aut}(\mathbb{K}^\times) \rtimes \mathrm{P}(V).$$

where $\mathrm{P}(V)$ *is the projective space associated to* V*.*

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Model theory of free buildings with no rank 3 residues of spherical type

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One of the key achievement from the work of Tits is the classification of all thick, irreducible, spherical buildings of rank at least 3. One crucial step in the proof of Tits' classification was the fact (proved in [19]) that all thick, irreducible, spherical buildings of rank at least 3 have the so-called Moufang property — namely if for every root α the associated root group U_α acts transitively on the set of all apartments of the building containing α. Crucially, the Moufang property does not hold in general for arbitrary Coxeter diagrams and for this reason, as observed already by Tits in [21, §1.6], similar classification results seem to be impossible in the general case of arbitrary buildings. Indeed, already in [20], Tits introduced a construction of free generalised n-gons (i.e., buildings of rank 2), which in contrast to the case studied in [22] are not Moufang, and do not admit a simple characterisation by algebraic invariants. More generally, the possibility of free constructions for larger classes of buildings was mentioned by Tits in [21], and later pursued by Ronan in [14]. In particular, Ronan showed that whenever Π is a Coxeter diagram with no rank 3 residues of spherical type, it is possible to define a "free construction" of buildings of type Π, which in particular admit any desired generalised n-gons as their rank 2 residues, provided their parameters fit together. As remarked by Ronan, his construction displays the fact that buildings with no rank 3 residues of spherical type – and thus in particular affine buildings of rank 3 – cannot be classified. In this talk, we will present our analysis of the the first-order theories of buildings obtained by such free constructions. As witnessed by the monograph [16] and by several recent publications, the model theory of geometric buildings has been an important source of interest in mathematical logic.

On the one hand, buildings have been investigated in geometric stability theory, especially in connection with the so-called *ample hierarchy* (originally introduced in [6,13]). In [4], Baudisch and Pillay introduced the theory of the 2-dimensional free pseudospace as an example of an ω-stable theory which is 2-ample and does not interpret a field. This construction was generalised by Tent in [17] to arbitrary dimensions, and she also showed that the free psudospace of dimension n is n-ample but not $n+1$-ample. Independently, the same result was obtained by Baudisch, Martin-Pizarro and Ziegler in [2]. Crucially, the prime models of the theory of the free psuedospaces are right-angled buildings, i.e. buildings whose associated Coxeter diagram Π contains only edges labelled by 2 and ∞. Relying on this observation, Baudisch, Martin-Pizarro and Ziegler also provided in [3] a thorough study of the model theory of *all* right-angled buildings with infinite residues, a class which contains all the free psudospaces but also several additional examples. In a different, but closely connected direction, Müller and Tent showed in [11] that the ample hierarchy is strict also with respect to some classes of geometries which are *not* right-angled. Crucially, however, these geometries have finite Morley rank and do not interpret a field, which by the results from [10] means they are not buildings.

On the other hand, there has been a considerable interest in the model-theoretic literature on generalised n-gons, which are the main examples of rank 2 buildings, see e.g. [5, 10, 15, 18].

Most recently, the model theory of generalised n-gons was investigated in the context of free constructions of rank 2 geometries, a topic which was brought into the model-theoretic forefront by Funk and Strambach in [7]. In [9], Hyttinen and Paolini considered the first-order theory of Hall's free projective planes (originally introduced in [8]), and they proved that their first-order theory is complete and strictly stable. In particular, they show that free projective planes of different rank are elementarily equivalent, thus proving an analogous of Sela's result on free groups in the context of projective planes. Later on in [1], Ammer and Tent extended the results from [9] to all generalised n-gons for all $n < \omega$, thereby providing a full-fledge study of Tits' free generalised n-gons from [20]. We notice that in both [9] and [1] the theory of the free generalised n-gons was axiomatised via the notion of *open configuration*, which is a first-order condition (originally identified by Hall in [8]) forcing a structure to omit some specific finite configurations, whose prototypical example in the projective context is the Fano plane.

Building on these prior works, in [12] we provided a general model-theoretic framework to study the first-order theory of all open incidence geometries of rank 2 considered in [7], such as generalised n-gons, Steiner systems, Benz planes and alike. As announced in [12], we continue our work in this article by considering geometries of higher rank, and most specifically Tits buildings. We thus focus on a specific subclass of the free buildings which can be obtained via Ronan's construction from [14], namely those whose rank 2 residues are *exactly* the free generalised n-gons introduced by Tits in [20] and studied in [1, 9].

Our approach is thus the following. Given a Coxeter diagrams Π, we introduce a notion of *open* Π*-geometry* and we define the associated first-order theory T^{o}_{Π} of open chamber systems of type Π with infinite residues . Under the assumption that Π has no rank 3 residues of spherical type, we then prove the crucial theorem that the Coxeter complex of type Π is open in this sense. Using this fact we use version of Ronan's construction to obtain a countable "free building". Crucially, this is both a model of T^{o}_{Π} and also satisfies a suitable universal mapping property relative to countable buildings of type Π. We then study at length the properties of the theory T^{o}_{Π} and we show in particular that T^{o}_{Π} is complete and stable. The following main theorem sums up all our main results.

Main Theorem. *For any Coxeter diagram* Π*, the theory* T^{o}_{Π} *of open chamber systems of type* Π *is complete and stable. Moreover, if* Π *has no rank 3 residues of spherical type, then there is, for all cardinal* $\kappa \geqslant \omega$*, a free building* B_{Π} *of size* κ*, which is both a model of* T^{o}_{Π} *and it satisfies a universal mapping property relative to buildings of type* Π*.*

Additionally, we also show that the requirement that Π has no rank 3 residues of spherical type is sharp, and we show that whenever Π contains a rank 3 residue of spherical type the associated Coxeter complex C_{Π} is not open, and thus T^{o}_{Π} does not have any model which is a building. Finally, we also we prove a number of additional model-theoretic results on T^{o}_{Π}: we characterise the elementary submodel relation $\preccurlyeq$ in terms of the open submodel relation $\leqslant_{\mathrm{o}}$, we characterise the forking-independence relation $\mathop{\smile\!\!\!\!\!\!|}$ in terms of canonical amalgamation, and we derive from results in [1, 3, 9] that T^{o}_{Π} is superstable only when Π is right-angled. To our knowledge, this work constitutes the first model-theoretic analysis of a class of buildings which is neither spherical nor right-angled.

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Parsing Correctness Criteria for Second-Order Multiplicative Linear Logic

1 Overview and Main Contributions

This work presents the first *parsing correctness criteria* for proof structures of second-order multiplicative linear logic MLL_2. Two classes of proof structures are considered: (1) *typed hypergraphs with explicit witnesses* for MLL_2, in which existential witnesses appear explicitly in the type labels; and (2) *standard typed hypergraphs* for MLL_2, in which witnesses are absent but *hidden proper variables* are tracked as additional data at each vertex. In both classes we defined a parsing reduction which identifies proof nets (i.e. hypergraphs which are the image of a proof tree).

The parsing reduction is proved **strongly normalising and confluent** in both cases, yielding a correct and efficient (quadratic-time) algorithm for deciding the correctness of MLL_2 proof structures. This contrasts with all previously known Danos–Regnier criteria for MLL_2, whose naive complexity is exponential in the number of links.

A further contribution concerns the *equivalence of proof trees* captured by each representation. Currently the equivalence is better understood on the proof trees of $\mathsf{MLL2}^{\mathrm{exp}}_{\mathrm{loc}}$, i.e. proof trees of MLL_2 whose existantial quantifiers have *explicit witnesses*. Two proof trees of $\mathsf{MLL2}^{\mathrm{exp}}_{\mathrm{loc}}$ are equal in their representation as typed hypergraphs with explicit witnesses, if and only if they are equal up-to *rule-commutation equivalence* (controlled by some global side conditions). Two proof trees of $\mathsf{MLL2}^{\mathrm{exp}}_{\mathrm{loc}}$ have the same representation as standard typed hypergraphs whenever they are equivalent not only up to rule commutations, but also up to a *rewitnessing* relation, analogous to the one appearing in Pistone's work on MLL_2 proof nets [6] and in Hughes's unification nets for $\mathsf{MLL1}$ [5].

Both of our parsing criteria reconstruct proof trees satisfying **Barendregt's convention**: the eigenvariable of every $\forall$-rule occurs free only above that rule. This property is not guaranteed by any existing parsing criterion for proof systems of (fragments of) linear logic with quantifiers, even in the first-order case, and is the direct reason why the parsing reductions we consider require a *global side condition* before certain rewriting steps.

2 Relation to Existing Work

Correctness criteria for MLL_2. Three lines of prior work address correctness for MLL_2 proof structures, all of them are Danos–Regnier criteria, making them of exponential complexity (at least naively):

- A *folklore* adaptation of Girard's first-order criterion [3] to second-order quantification.
- Strassburger's *proof graphs* [7, 8], which use boxes and handle units.
- Pistone's *G-nets* [6], which uses a box-free notion of proof structures with a Danos-Regnier criterion. The resulting proof structures are similar to Girard proof structures for first order adapted to second order.

Furthermore, none of these works formally identifies the equivalence relation on proof trees captured by equality of proof structures. The present work attempts to fill this gap aswell, two

proof trees with explicit exists rules are equal after desequentialisation if and only if they are equal up-to some rule commutations.

Parsing criteria with quantifiers. For some fragments of linear logic with first-order quantifiers, various parsing (also called coalescence) criteria already exist: for MLL1 [5], for purely additive linear logic with first-order quantifiers [4], and for an extension of $MALL1$ [1]. All of these use only *local* side conditions and none reconstructs only proof trees satisfying Barendregt's convention. The present criterion is the first parsing criterion with second-order quantifiers and the first to enforce a global side condition in the rewriting, allowing to reconstruct only proof trees satisfying Barendregt's convention. This is discussed more precisely in the next section.

3 Proof-Induced Identity of Variables

We identify and formulate a conceptual problem which must be addressed when dealing with quantifiers that we deem the *proof-induced identity* of variables: when do two occurrences of a variable in a proof tree refer to the same variable? We identify two solutions to the problem, the *notational approach* and the *constraint-based approach*. All the currently existing parsing (coalescence) correctness criterion for linear logic with quantifiers rely on the constaint-based approach, ours is the first to take the notational approach.

Notational approach (this work). Two occurrences of propositional variables are identified as soon as they share the same name. Consequently, whenever a propositional variable X appears in two distinct places in a proof tree, both occurrences refer to exactly the same variable. Coherence under substitution and cut-elimination then requires the global *Barendregt's convention* [2]: the eigenvariable of a $\forall$-rule may occur free only strictly above that rule. Naive local rule-commutations can violate this convention because a subtree containing an eigenvariable X may move below the $\forall$-rule that owns the eigenvariable X. As a consequence rule-commutation equivalence imposes a global side condition before permuting a ($\forall/\otimes$) or ($\forall$/cut) pair. Concrete examples show that ignoring this condition leads to cut-elimination steps that can turn a correct $\forall$-rule into an incorrect one.

Examples. Local rule commutations may break Barendregts convention:

$$\dfrac{\dfrac{W,Y}{W,\forall ZZ}\forall \quad \dfrac{W^\perp}{\exists UU^\perp}\exists}{\dfrac{W}{\forall XX}\forall}\,\mathsf{cut} \quad\sim\quad \dfrac{\dfrac{\dfrac{W,Y}{W,\forall ZZ}\forall}{\forall XX,\forall ZZ}\forall \quad \dfrac{W^\perp}{\exists UU^\perp}\exists}{\forall XX}\,\mathsf{cut} \quad\sim\quad \dfrac{\dfrac{\dfrac{W,Y}{\forall XX,Y}\forall}{\forall XX,\forall ZZ}\forall \quad \dfrac{W^\perp}{\exists UU^\perp}\exists}{\forall XX}\,\mathsf{cut}$$

A proof tree which does not satisfy Barendregts convention, can turn a correct $\forall$–rule into an incorrect one during cut–elimination:

$$\dfrac{\dfrac{\dfrac{W,Y}{\forall XX,Y}\forall}{\forall XX,\forall ZZ}\forall \quad \dfrac{W^\perp}{\exists UU^\perp}\exists}{\forall XX}\,\mathsf{cut} \quad\to\quad \dfrac{\dfrac{W,W}{\forall XX,W}\forall \quad W^\perp}{\forall XX}\,\mathsf{cut}$$

Constraint-based approach. Two occurrences of a variable are identified only when a structural constraint (i.e. an axiom link, a cut, or an exists-rule witness) forces them to coincide. This renders Barendregt's convention unnecessary and allows purely local rule commutations, at the cost of a richer, constraint-annotated syntax. This is the approach of Hughes's unification nets [5] for MLL1, and appears more natural for untyped MLL_2 proof structures.

4 Future Directions

Geometric criterion for MLL_2. The parsing criterion should be equivalent to the Danos–Regnier criterion of [3, 6]. This would formally relate the two frameworks and yield a new proof of correctness for the Danos–Regnier criterion in the MLL_2 case. To show the equivalence of the two criterion a proof plan consists in proving the following: (1) non-parseable proof structures fail the switching condition, (2) parsing reductions preserve the switching condition, (3) parsing anti-reductions preserve the switching condition.

Untyped parsing criterion for MLL_2. Extending the criterion to untyped MLL_2 proof structures requires an extension of our current framework. In particular, the correctness criterion would consist of two procedures: *parseability*, which ensures that the proof structure admits a valid tree structure (in particular for quantifiers, a $\forall$-link is parsed only after all the necessary $\exists$-link have parsed), and *typeability*, which ensures that the proof structure can be properly typed.

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Post Completeness in Conditional Logic

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1 Introduction and Motivation

A systematic study into the structure of the lattice of conditional logics is notably missing in the literature. In this work, we begin filling this gap by investigating the co-atoms in this lattice, i.e., maximal consistent logics, or equivalently, Post complete logics. Specifically, we characterize several Post complete conditional logics, determine the number of such logics within relevant sublattices, and draw broader consequences from these findings. Our results can be summarized as follows:

1. We determine the number of Post complete regular (and normal) conditional logics;
2. We axiomatically characterize the Post complete regular (and normal) conditional logics;
3. As a consequence, we prove that every consistent regular logic has a frame, i.e., it is sound with respect to a frame.
4. We show that there are uncountably many Post complete seminormal conditional logics.

To establish our framework, we define the formal language $\mathcal{L}_>$ over a denumerable set of variables $Var = \{p, q, r, \dots\}$ and the connectives $\bot, \neg, \wedge, \vee, >$. Here, $>$ is a binary connective representing a conditional, where $\varphi > \psi$ reads *if* φ *then* ψ. Derived connectives (material implication $\rightarrow$, material equivalence $\leftrightarrow$, and $\top$) are defined as usual. $Fm_>$ denotes the set of formulas in this language, defined according to the standard formation rules. Lowercase Greek letters $\varphi, \psi, \delta, \dots$ denote arbitrary formulas in $Fm_>$. In this context, a logic is defined as follows:

Definition 1. *A conditional logic* L *is a set of formulas* $\mathsf{L} \subseteq Fm_>$ *that contains all classical propositional tautologies and is closed under modus ponens (for material implication* $\rightarrow$*) and uniform substitution.*

Henceforth, by a *logic*, we simply mean a conditional logic in the language $\mathcal{L}_>$. Derivability in a logic L is denoted by the symbol $\vdash_{\mathsf{L}}$; $\vdash_{\mathsf{L}} \varphi$ means that φ is a theorem of L (i.e., $\varphi \in \mathsf{L}$). A conditional logic L is consistent iff $\nvdash_{\mathsf{L}} \bot$. From Definition 1, it is clear that the smallest conditional logic is simply classical propositional logic (PL) as formulated in $Fm_>$. The greatest conditional logic is the trivial logic, that is, $Fm_>$ itself. More interesting systems are obtained

by augmenting PL with axioms or rules such as:

$$(\varphi > (\psi \wedge \theta)) \rightarrow ((\varphi > \psi) \wedge (\varphi > \theta)) \quad \text{(CM)}$$

$$((\varphi > \psi) \wedge (\varphi > \theta)) \rightarrow (\varphi > (\psi \wedge \theta)) \quad \text{(CC)}$$

$$\varphi > \top \quad \text{(CN)}$$

$$\frac{\vdash \varphi \leftrightarrow \psi}{\vdash (\varphi > \theta) \leftrightarrow (\psi > \theta)} \quad \text{(RCEA)}$$

$$\frac{\vdash \varphi \leftrightarrow \psi}{\vdash (\theta > \varphi) \leftrightarrow (\theta > \psi)} \quad \text{(RCEC)}$$

A conditional logic is semiregular (regular) if it contains (CM) and (CC) and is closed under (RCEC) (and (RCEA)). A conditional logic is seminormal (normal) if it is semiregular (regular) and contains (CN). The smallest semiregular, seminormal, regular, and normal conditional logics are called Cr, Ck, CR, and CK, respectively.

A logic L is an extension of a logic K if $\mathsf{K} \subseteq \mathsf{L}$. It is clear that the extensions of any conditional logic L form a complete lattice ordered by inclusion ($\subseteq$), in which the intersection ($\cap$) serves as the meet, and the closure of the union under modus ponens and uniform substitution ($+$) serves as the join. By $\mathsf{L} + \varphi$, we mean the smallest extension of L containing the axiom φ; we may refer to $\mathsf{L} + \varphi$ as the axiomatic extension of L by φ.

In our investigation, we will focus on the lattice of extensions of Cr, the smallest semiregular conditional logic, which captures some of the minimal logical properties that a conditional connective should satisfy. The lattice of extensions of Cr contains most, if not all, of the conditional logics studied in the literature. In particular, it contains the logic CK, which corresponds to Chellas's [1] Basic Conditional Logic, as well as the entire family of variably strict conditional logics introduced by Lewis [3], all of which are proper extensions of CK. In general, the extensions of Cr include all the conditional logics listed in [2]. For this reason, we may safely refer to the lattice of extensions of Cr simply as the lattice of conditional logics. In what follows, we will study the top layer of this lattice. That is, we will characterize the coatomic logics (namely, the Post complete conditional logics) within several relevant subfamilies of semiregular conditional logics. In particular, we will focus on regular, normal, and seminormal conditional logics.

2 Post complete Logics

The consistent logics that are maximal with respect to set inclusion within the lattice of conditional logics are the *Post complete* logics, defined as follows:

Definition 2. *A conditional logic L is Post complete if it is consistent but has no consistent proper extensions (equivalently, whose only proper extension is $Fm_{>}$).*

Clearly, for a logic L, its (regular/seminormal/normal) Post complete extensions are those (regular/seminormal/normal) Post complete logics L' that contain L, i.e., $\mathsf{L} \subseteq \mathsf{L}'$.

We characterize the Post complete logics within specific sublattices of conditional logics, paralleling Segerberg's work and techniques [6] on the lattice of modal logics. Our main result for regular (and consequently, normal) conditional logics is established by the following theorem.

Theorem 1. *There are exactly 9 Post complete regular logics. These are the axiomatic exten-*

sions of CR *via each of the following axioms, respectively*

$$(\varphi > \psi) \leftrightarrow \top \quad \text{(TOP)}$$
$$(\varphi > \psi) \leftrightarrow (\varphi \vee \psi) \quad \text{(DISJ)}$$
$$(\varphi > \psi) \leftrightarrow \varphi \quad \text{(ANT)}$$
$$(\varphi > \psi) \leftrightarrow (\varphi \rightarrow \psi) \quad \text{(IMP)}$$
$$(\varphi > \psi) \leftrightarrow \psi \quad \text{(CON)}$$
$$(\varphi > \psi) \leftrightarrow (\varphi \wedge \psi) \quad \text{(CONJ)}$$
$$(\varphi > \psi) \leftrightarrow \neg\varphi \quad \text{(NANT)}$$
$$(\varphi > \psi) \leftrightarrow (\neg\varphi \wedge \psi) \quad \text{(NCONJ)}$$
$$(\varphi > \psi) \leftrightarrow \bot \quad \text{(BOT)}$$

As a corollary of the above, we obtain exactly 4 Post complete normal conditional logics: CR + (TOP), CR + (DISJ), CR + (IMP), and CR + (CON). Interestingly, in each of these 9 Post complete logics, the conditional connective collapses into a combination of purely Boolean connectives. Consequently, roughly speaking, these Post complete logics are essentially equivalent to classical propositional logic, where the $>$ operator plays no distinctive role.

The following theorem constitutes another significant result.

Theorem 2. *Every consistent regular conditional logic* L *is contained in at least one of the nine Post complete regular conditional logics.*

Beyond providing insight into the structure of the lattice of conditional logics, the results above allow us to derive significant consequences for their semantics. To this end, we now introduce neighborhood frames.

Definition 3. *A neighborhood frame is a tuple* $\mathcal{F} = \langle W, \mathcal{N} \rangle$ *where* $W \neq \emptyset$ *is a set of worlds and* $\mathcal{N} : W \times \mathcal{P}(W) \rightarrow \mathcal{P}(\mathcal{P}(W))$ *is a function indexed by worlds and propositions satisfying the following constraint:*

$$(X \cap Y) \in \mathcal{N}(w, Z) \text{ if and only if } X, Y \in \mathcal{N}(w, Z) \quad \text{(C)}$$

A neighborhood model $\mathcal{M} = \langle W, \mathcal{N}, [\cdot] \rangle$ is obtained by equipping a neighborhood frame with a valuation function $[\cdot] : Var \rightarrow \mathcal{P}(W)$. This function is extended to complex formulas inductively; for Boolean connectives, the clauses are standard, while for the conditional we have:

$$[\varphi > \psi] = \{w \in W \mid [\psi] \in \mathcal{N}(w, [\varphi])\}$$

These neighborhood models, introduced by Chellas [1], generalize the selection function-based models for conditional logics in the tradition of Stalnaker [7] and Lewis [3]. Specific classes of neighborhood frames provide a semantics for regular and normal conditional logics. In particular, we say that a conditional logic L has a frame if there exists a neighborhood frame $\mathcal{F} = \langle W, \mathcal{N} \rangle$ such that for every model $\mathcal{M} = \langle W, \mathcal{N}, [\cdot] \rangle$ based on $\mathcal{F}$, $[\varphi] = W$ for all $\varphi \in$ L. In other words, L has a frame if it is sound with respect to some neighborhood frame. It is not hard to show that that every Post complete regular logic has a frame; consequently, in conjunction with Theorem 2, we obtain the following:

Theorem 3. *Every consistent regular (and normal) conditional logic has a frame.*

We now turn to the sublattice of seminormal conditional logics. Unlike the previous cases, the number of Post complete logics in this subfamily is not finite. Indeed, by extending Segerberg's techniques for modal logic to the conditional case, we establish the following:

Theorem 4. *There are* $2^{\aleph_0}$ *many Post complete seminormal conditional logics.*

By abstracting from specific classes of logics and considering general conditional logics without the requirement of (semi)normality or (semi)regularity, we yield the following:

Theorem 5. CK *has* $2^{\aleph_0}$ *many Post complete extensions.*

This result parallels Segerberg's [6] proof that the basic modal logic K has uncountably many Post complete extensions (including non-normal ones).

3 Conclusions and Future Work

To the best of our knowledge, this work represents the first systematic investigation into the structure of the lattice of conditional logics. By characterizing the Post complete logics within relevant subfamilies, we provide a map of the top layer of this lattice. These results serve as a foundation for further exploration of the internal structure of conditional logics.

A promising future direction involves adopting an algebraic perspective by studying the dually isomorphic lattice of subvarieties within the equivalent algebraic semantics of conditional logics. For the sublattice of normal conditional logics, this could be achieved by extending the classes of algebras presented in [5]. Such an extension may allow for the characterization of, for instance, splitting pairs within the family of normal conditional logics. However, for broader sublattices, such as seminormal conditional logics, standard algebraic techniques may not directly apply, as some seminormal logics are not closed under uniform substitution. In such cases, one may explore alternative approaches based on recent work regarding the algebraizability of logics lacking uniform substitution [4].

Finally, given the parallels between our results and those in modal logic, the relationship between the lattice of conditional logics and multimodal logics warrants further study. We conjecture that the lattice of multimodal logics with infinitely many modalities may be embeddable within the lattice of conditional logics, where the latter exhibits a significantly higher degree of complexity and fine-grainedness.

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Quasi-topos semantics

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Talk proposal for AILA 2026

Category theory provides a broad generalization of the standard set-theoretic semantics of algebraic structures and logical theories. Indeed, categories can be used as universes in which languages from a wide range of formalisms – from first-order theories to classical set theories can be modeled. In these semantics, the standard soundness and completeness theorems hold; in particular, completeness is shown by constructing, for a theory T, a canonical model in a *syntactic category* $\mathrm{Synt}(T)$, such that the following equivalence holds:

$$T \vdash \varphi \Leftrightarrow \mathrm{Synt}(T) \models \varphi$$

Moreover, for most categorical semantics of logical theories, a strengthening of the completeness theorem holds as well: one can associate to each category $\mathcal{C}$ an *internal theory* $\mathrm{Int}(\mathcal{C})$, which captures all the constructions that can be performed in $\mathcal{C}$, so that $\mathrm{Synt}(\mathrm{Int}(\mathcal{C}))$ is categorically equivalent to $\mathcal{C}$. Lawvere-Tierney elementary toposes provide a notable example of categories whose internal theories can be used to formalize mathematics, especially constructive mathematics.

In [Mai05] it was shown that, for elementary toposes and various other categorical structures, one can associate an internal calculus with dependent types for which the assignments

$$T \mapsto \mathrm{Synt}(T) \qquad \mathcal{C} \mapsto \mathrm{Int}(\mathcal{C})$$

give rise to a tighter correspondence between syntax and semantics, amounting to a form categorical equivalence.

In this talk we will focus on a generalization of toposes, called quasi-toposes, introduced by Jacques Penon in the 70s [Pen76]. Quasi-toposes are obtained by relativizing the definition of an elementary topos to a general notion of subobjects, whereas in an elementary topos this coincides with the collection of (equivalence classes of) monomorphisms. Logically, this is reflected in the fact that quasi-toposes do not automatically validate the following Axiom of Unique Choice.

$$\forall x \in A\, \exists! y \in B.R(x,y) \Rightarrow \exists f : A \to B\, \forall x \in A.R(x, f(x))$$

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We will show that quasi-toposes can be viewed as the semantics of a suitable dependent type theory obtained as an impredicative version of the extensional level of the Minimalist Foundation in [Mai09] designed according to the principles in [MS05] and that our dependent type theory provides an internal language for quasi-toposes in the sense of [Mai05].

As a relevant application of this result, we show that the type theory underlying the proof assistant Rocq, known as the Calculus of Constructions [CH88], can be modeled in the constructive arithmetical, solid quasi-toposes built in [MST26] by using the technique in [MS24]. These models simultaneously validate some computable principles adopted in Russian Constructivism, including a restricted form of Church's Thesis, called Type-theoretic Church Thesis, and all continuity principles adopted in Brouwerian intuitionism. This provides a reconciliation between Russian Constructivism and Brouwerian intuitionism, which we show to be simultaneously inconsistent within the internal language of any topos by a well-known Kleene argument.

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Left and Right Effective Symmetries on Baire and Cantor Space

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The symmetric group $\mathrm{Sym}(\omega)$ acts on Baire space ω^ω and Cantor space 2^ω in two canonical ways: on the left, by relabeling values, and on the right, by reindexing positions. Restricting to effective subgroups—most notably the computable permutations and the fully primitive recursive ones—yields natural countable Borel equivalence relations. Writing E_L^c and E_R^c for the orbit equivalence relations of the computable left and right actions, $E_R^c(2^\omega)$ is exactly recursive isomorphism, equivalently 1-equivalence, on subsets of ω. Classical results show that $E_R^c(\omega^\omega)$ is a universal countable Borel equivalence relation [2]; universality has been progressively pushed down to $E_R^c(k^\omega)$ for $k \geq 5$ [1] and then $k \geq 3$ [4]; the case $k = 2$ remains a long-standing open problem [4]. By contrast, the corresponding left action—equally natural from a group-theoretic standpoint—has, to our knowledge, not been studied systematically.

In this talk, we report on a recent project that begins to fill this gap, by comparing the orbit equivalence relations arising from the left and right actions of the computable and fully primitive recursive symmetric groups on 2^ω and ω^ω under three main notions of reducibility: computable, continuous, and Baire class 1. The resulting picture is unexpectedly rich: these notions induce qualitatively different landscapes, with collapses appearing as the reducibility class is enlarged. At the computable level, the structure is rigid. While $E_L^c(2^\omega)$ is easily seen to be smooth, on Baire space the two actions are computably incomparable, with E_0 reducing to $E_L^c(\omega^\omega)$ but not to $E_R^c(\omega^\omega)$; moreover, neither relation is computably reducible to any effectively Π_3^0 equivalence relation. Weakening the notion of reduction substantially alters the picture: on Baire space, the computable left action becomes Baire class 1 reducible to the computable right action.

The proofs draw on tools from both computability theory and descriptive set theory. The work thus fits into a broader recent interest in finer notions of effective reducibility for equivalence relations [5, 3], and isolates a natural family in which the two perspectives genuinely interact.

This is joint work with Jun Le Goh, Heer Tern Koh, and Keng Meng Ng.

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AUTOMORPHISM GROUPS OF NON-ARCHIMEDEAN GROUPS

ANDRÉ NIES AND PHILIPP SCHLICHT

A Polish group G is called *non-Archimedean (nA)* if G has a countable basis $\mathcal{S}_G = \{U_n : n \in \omega\}$ of neighbourhoods of the identity that consists of open subgroups. Such a group G is topologically isomorphic to a closed subgroup of S_∞, the group of permutations of $\mathbb{N}$ with the topology of pointwise convergence. (To see this, one lets G act from the left on the set of left cosets of subgroups in $\mathcal{S}_G$.) Conversely, each closed subgroup of S_∞ is nA, via taking as U_n the permutations in G that fix each $i < n$. One verifies that the closed subgroups of S_∞ are precisely the automorphism groups of structures with domain ω. The class of nA groups enjoys several permanence properties: closed subgroups, quotients by closed normal subgroups, and countable Cartesian products of nA groups are again nA.

We focus on two groups derived from a nA group G: $\mathrm{Aut}(G)$, the group of *continuous* automorphisms of G, and $\mathrm{Out}(G)$, the quotient of $\mathrm{Aut}(G)$ by its normal subgroup of inner automorphisms. One says that a closed subgroup G of S_∞ is *oligomorphic* if for each $n \in \omega$, its action on ω^n has only finitely many orbits. We will in particular address $\mathrm{Aut}(G)$ and $\mathrm{Out}(G)$ for oligomorphic G.

There are two interrelated parts.

(A) The natural action of $\mathrm{Aut}(G)$ on G is given by $\alpha \cdot g = \alpha(g)$ for $\alpha \in \mathrm{Aut}(G)$ and $g \in G$. We provide a sufficient criterion when $\mathrm{Aut}(G)$ has a compatible Polish topology that makes this action continuous: *G has a countable neighbourhood basis $\mathcal{S}_G$ of the neutral element consisting of open subgroups that can be chosen invariant under this action.* In this case, $\mathrm{Aut}(G)$ is nA itself.

(B) The class of oligomorphic groups satisfies the criterion; also, $\mathrm{Inn}(G)$ is closed in $\mathrm{Aut}(G)$. So $\mathrm{Out}(G)$ is a Polish group. We study it through the model-theoretic notion of bi-interpretations, and use this to give a model-theoretic proof of the result in [3] that $\mathrm{Out}(G)$ is totally disconnected and locally compact.

The study of $\mathrm{Out}(G)$ is motivated in part by the question whether the isomorphism relation between oligomorphic groups is smooth in the sense of Borel reducibility. An upper bound on this relation is known by [4]: it is essentially countable. However the precise complexity, first asked in [2], remains unknown. Certain subclasses are known to be smooth [3], such as the automorphism groups of ω-cateogrical structures without algebraicity.

In contrast, the conjugacy relation on the Borel space of oligomorphic groups is smooth [4]. Towards answering the question, it is thus useful to know whether an isomorphism between two permutation groups G and H is induced by a conjugation with a permutation of $\mathbb{N}$. For a single permutation group G, $\mathrm{Out}(G)$ is trivial iff every continuous automorphism of G is given by conjugating with a permutation of $\mathbb{N}$ in G.

We first define Roelcke precompact Polish groups in the nA case.

Definition 0.1. A non-Archimedean group G is *Roelcke precompact* if each open subgroup has only finitely many double cosets.

(A) When is $\mathrm{Aut}(G)$ non-Archimedean? The main result for this part is:

Theorem 0.2. *Suppose a nA group satisfies the criterion in (A) above. Then there is a copy $\widehat{G}$ of G as a closed subgroup of S_∞ such that the group $\mathrm{Aut}(G)$ is isomorphic to the Polish group that is given as the normaliser of $\widehat{G}$ in S_∞ by its centraliser in S_∞. The induced topology on $\mathrm{Aut}(G)$ is the unique Polish topology that makes the action on G continuous.*

The criterion implies that $\mathrm{Inn}(G)$ is a Borel subgroup of $\mathrm{Aut}(G)$ that is Polishable. As an application of the criterion, $\mathrm{Aut}(G)$ is itself nA for several natural classes of nA groups. This includes the locally Roelcke precompact groups, where the neighbourhood basis in (A) consists of the Roelcke precompact open subgroups of Definition 0.1. We note that if G is Roelcke precompact, then $\mathrm{Inn}(G)$ is in fact a closed subgroup of $\mathrm{Aut}(G)$ [3, Th. 2.7].

(B) Oligomorphic groups and bi-interpretations. Each oligomorphic group is Roelcke precompact by [5, Th. 2.4]. If G is oligomorphic, the outer automorphism group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is t.d.l.c. by [3, Th. 3.10]. We wish to describe $\mathrm{Out}(G)$ based on the theory of any ω-categorical structure that has G as its automorphism group. For an ω-categorical theory T, let $\mathcal{B}(T)$ be the group of self-interpretations of T that have an inverse, all up to definable bijections between sorts of T^{eq}. The following reproves the result that $\mathrm{Out}(G)$ is t.d.l.c. using model theory, and gives a model theoretic description of $\mathrm{Out}(G)$ based only on the theory of the underlying structure.

Theorem 0.3. *Let G be an oligomorphic group, and let T be the elementary theory of a structure M such that $G = \mathrm{Aut}(M)$.*

(i) The group $\mathcal{B}(T)$ is totally disconnected, locally compact.
(ii) $\mathcal{B}(T)$ is topologically isomorphic to $\mathrm{Out}(G)$.

Remark 0.4. Outer automorphism groups were introduced to study finite groups. For countable groups, outer automorphism groups are important in the study of mapping class groups of surfaces. Dehn, Nielsen and Baer showed that the extended mapping class group $\mathrm{MCG}^+(S)$ of a compact closed orientable surface S of genus $g \geq 1$ is isomorphic to $\mathrm{Out}(\pi(S))$. Here $\mathrm{MCG}^+(S)$ denotes the quotient of the group of orientation preserving homeomorphisms of S by the connected component of the identity, and $\pi(S)$ is the fundamental group [1, Theorem 8.1].

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1 Introduction

A century ago, Polish mathematicians Stefan Banach and Alfred Tarski managed to prove a result that, in some way, changed our view of mathematics forever. Their renowned theorem, published in [BT24], shows that it is possible to divide the closed unit ball $\mathbb{D}^3 \subseteq \mathbb{R}^3$ into finitely many pieces and reassemble them to form two balls of the same size, hence duplicating the total volume, something that in "real life" looks impossible.

Banach and Tarski's proof made use of the axiom of choice (AC), whose content is not constructive. The original goal of the two mathematicians was to provide supporting arguments to their aversion to AC, showing a paradoxical behaviour in its consequences. Indeed, there are alternative settings where mathematics can be developed, like the Solovay model introduced later in [Sol70], where the axiom of choice fails and every set of reals is Lebesgue measurable, hence the ball duplication result also fails.

Though, Banach and Tarski's result did not have the consequences they expected. Indeed, today the vast majority of the mathematical community accepts the axiom of choice and its consequences, considering them simply as counterintuitive but not as paradoxical.

Returning to [BT24], no estimate on the number of pieces necessary for the decomposition was given therein. In the following years many mathematicians tried to find an upper bound: John von Neumann claimed in [Neu29] that nine pieces suffice. Later, in [Sie45], Wacław Sierpiński proved that eight are enough. Eventually, in [Rob47], Raphael Robinson settled the question showing that five pieces are the minimal number.

In his proof, Robinson showed that the same construction for the unit sphere $\mathbb{S}^2$ (the boundary of $\mathbb{D}^3$) requires four pieces. Starting from such a

minimal decomposition he was able to construct one for $\mathbb{D}^3$ with only five elements, one of which is a singleton, and he also showed that with less than five pieces it is impossible to reach the same result.

It is clear that, iterating this construction, one can start with a unit ball, divide it into finitely many pieces, and reassemble them into n balls of the same size, for all natural numbers n. A question that now naturally arises is the following: what is the minimal quantity of pieces needed to produce n unit balls starting with one?

This work is motivated by the authors' aim of answering the latter question. We started by looking whether it had already been discussed somewhere, and we only found a Mathematics StackExchange post from twelve years ago in which an anonymous user posed the same problem.

A tentative answer was suggested by Andrés Caicedo in [Cai13], where he claims that $5n - 2$ pieces are surely sufficient and states that "it would be tempting to say that" the answer is $2n + 1$, probably based on the idea that for the sphere $2n$ pieces are the correct value, and then one more piece is needed to pass from $\mathbb{S}^2$ to $\mathbb{D}^3$, as done by Robinson in the $n = 2$ case.

2 Results

In my talk I am going to show that, indeed, the necessary number of pieces for dividing $\mathbb{S}^2$ and reassembling it into n copies is $2n$, while, for $\mathbb{D}^3$, we prove that the value of the optimal decomposition is $3n - 1$ pieces, $n - 1$ of which can be taken as singletons.

Concerning our contribution, we begin our proof with a group-theoretic result (Theorem 2.5), that builds upon the classical proof for the $n = 2$ case (Theorem 2.2). In this regard see, for instance, the monograph [Wag85] or its updated edition [TW16].

Theorem 2.5. *For each $\omega \in \mathbb{F}_2$, there is a partition of $\mathbb{F}_2$ in $2n$-many subsets*

$$\mathbb{F}_2 = A_0 \sqcup B_0 \sqcup \ldots \sqcup A_{n-1} \sqcup B_{n-1}$$

with ε and ω in the same element of the partition, and such that for all $i < n$

$$\gamma_i(B_i) = \mathbb{F}_2 - A_i.$$

Theorem 2.2. *For each $\omega \in \mathbb{F}_2$, there is a partition of $\mathbb{F}_2$ in four subsets*

$$\mathbb{F}_2 = A_0 \sqcup B_0 \sqcup A_1 \sqcup B_1$$

such that ε and ω belong to the same element of the partition, and

$$\sigma(B_0) = \mathbb{F}_2 - A_0; \quad \tau(B_1) = \mathbb{F}_2 - A_1.$$

We point out that each proof of Theorem 2.2 makes use of a suitable partition of $\mathbb{F}_2$ into four subsets. Each such partition is a slight variant of that formed by the sets "words beginning with σ", "words beginning with τ", "words beginning with σ^{-1}" and "words beginning with τ^{-1}", where σ and τ are the generators of $\mathbb{F}_2$.

In the $n > 2$ case, the main issue is to partition $\mathbb{F}_2$ into $2n$ subsets. As $\mathbb{F}_2$ is two-generated, finding a decomposition into $2n$ sets that is suitable for our later purposes is not completely trivial.

Another significant complication in the $n > 2$ case arises in the proof of Theorem 4.4 below. Therein we need to single out a word ω in a very careful manner. Notice that, in the $n = 2$ case, this is not needed because of the high symmetry of the four sets forming the $\mathbb{F}_2$ partition whose existence is claimed in Theorem 2.2. Such symmetry is then used throughout the whole proof.

Theorem 4.4. *If the action $\mathbb{F}_2 \curvearrowright \Omega$ is locally abelian, then Ω is $(n, 2n)$-paradoxical with respect to this action.*

Our final results are then the following:

Corollary 4.8 (Generalised Banach-Tarski for $\mathbb{S}^2$)**.** *With respect to $\mathrm{SO}(3) \curvearrowright \mathbb{R}^3$, the sphere $\mathbb{S}^2$ is $(n, 2n)$-paradoxical and not (n, r)-paradoxical for $r < 2n$.*

Corollary 5.6 (Generalised Banach-Tarski for $\mathbb{D}^3$)**.** *With respect to $\mathrm{SE}(3) \curvearrowright \mathbb{R}^3$, the ball $\mathbb{D}^3$ is $(n, 3n-1)$-paradoxical and is not (n, r)-paradoxical for $r < 3n - 1$.*

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A completeness theorem in proof-theoretic semantics

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1 Introduction

With respect to the introduction and elimination rules of natural deduction, Gentzen [3] made the following remark. *The introductions represent, as it were, the 'definitions' of the symbols concerned, and the eliminations are no more, in the final analysis, than the consequences of these definitions.* By developing Gentzen's remark, Prawitz [13] defined the validity of proof-structures in natural deduction. A proof-structure of an atomic formula (atom) is valid if it reduces to an atomic structure belonging to a predefined atomic base. A proof-structure of a complex formula is then valid, if it reduces to a canonical form ending with an introduction rule, and if its immediate substructures are valid. Schroeder-Heister has extensively analyzed the proof-theoretic validity and developed what is known as "proof-theoretic semantics." See, for example, [16, 11]. Since Prawitz conjectured the completeness of intuitionistic logic with respect to proof-theoretic semantics, various conditions on proof-theoretic semantics have been investigated with the aim of solving the conjecture. See [9, 12, 10]. However, these completeness and incompleteness results are mainly obtained by focusing on logical consequence among formulas by abstracting away from proof-structures.

By returning to Prawitz's original framework, that is, by considering the validity of proof-structures, Piccolomini d'Aragona classified proof-theoretic semantics according to the following viewpoints: Whether or not extensions of atomic bases are considered (monotonic or non-monotonic) in interpreting implication, as in Kripke semantics; Whether connectives are interpreted on the basis of natural deduction introduction rules or elimination rules (introduction-based or elimination-based). He then established various incompleteness results within these frameworks [4, 5, 6, 7]. Very recently, Piccolomini d'Aragona and Prawitz [8] proved that intuitionistic propositional logic is incomplete with respect to non-monotonic, introduction-based proof-theoretic semantics.

One of the essential causes of the incompleteness results lies in the interpretation of open proof-structures (proof-terms). An open structure $t[x]$ of A, with a free term-variable x of B, is valid if and only if, for every valid closed structure s of B, the substitution $t[x := s]$ is valid. If there exists no valid closed structure of B, then $t[x]$ of A is vacuously valid. We address this difficulty by introducing term-constants, thereby preventing the non-existence of closed structure of each formula. However, from the viewpoint of natural deduction, these term-constants represent non-logical axioms; hence, by introducing constants for all formulas, every formula would become derivable. Thus, we distinguish derivable structures containing constants from those without constants, and we regard a derivable structure with constants as a proof in the process of being constructed, which requires a further proof. Accordingly, in our proof-theoretic semantics, we introduce the notion of quasi validity (q-validity) for structures that may contain constants, and we apply the notion of validity to structures without constants.

The aim of this study is to prove the completeness of intuitionistic propositional logic with respect to Prawitz's proof-theoretic validity within the non-monotonic, elimination-based framework. As an intuitionistic natural deduction system, we employ atomic second-order intuitionistic propositional logic $\mathsf{IL}^2\mathsf{at}$, which is essentially the same system as atomic polymorphism

$\mathbf{F_{at}}$ [2, 1]. Although the $\forall$-elimination rule in this system restricts universal instantiation to an atom, the other intuitionistic connectives $\wedge, \vee, \bot$ (and $\exists$) are definable using only $\to$ and atomic $\forall$ [1]. Furthermore, atomic $\forall$ in $\mathsf{IL}^2\mathsf{at}$ can be semantically interpreted in essentially the same way as the standard first-order $\forall$. Thus, we investigate $\mathsf{IL}^2\mathsf{at}$ as a system that includes intuitionistic propositional logic while avoiding the difficulties associated with $\vee$ (and $\exists$), as well as the full complexity of second-order logic.

2 $\mathsf{IL}^2\mathsf{at}$

We define our *atomic second-order intuitionistic propositional logic* $\mathsf{IL}^2\mathsf{at}$.

Definition 1. Formulas of $\mathsf{IL}^2\mathsf{at}$ are defined as follows, where X is an atom (propositional variable).

$$A, B ::= \quad X \quad | \quad A \to B \quad | \quad \forall X.A$$

In addition to the usual λ-terms, we introduce the term-constant c^A for every A.

Definition 2. Terms are defined as follows.

$$t, s ::= \; x \; | \; c^A \; | \; (\lambda x.t) \; | \; (ts) \; | \; (\Lambda X.t) \; | \; (tX)$$

We omit parentheses as usual. We denote term-variables by small letters $x, y, z, \ldots$, and propositional variables (atoms) by capital letters $X, Y, Z, \ldots$. By $FV(t)$, we denote the set of term-variables as well as propositional variables that freely appear in term t. By $tFV(t)$, we denote the set of free term-variables in t. A term t is **closed** if no term-variable appears freely in t, i.e., $tFV(t) = \emptyset$; otherwise, t is **open**. Substitutions for free term-variables, $t[x := s]$, and for free propositional variables, $t[X := A]$ and $A[X := B]$, are defined as usual. We assume the usual α-equivalence for both terms and formulas.

Inference rules of $\mathsf{IL}^2\mathsf{at}$ are the usual second-order natural deduction rules except for $\forall$-elimination rule, whose instantiation is restricted to an atom.

Definition 3 ($\mathsf{IL}^2\mathsf{at}$)**.** A statement is of the form $t : A$ with a term t and a formula A. A **context** (or **assumptions**) is a finite set of statements such as $x_1 : A_1, \ldots, x_n : A_n$ where all $x_1, \ldots, x_n$ are distinct term-variables. We write $\Gamma, \Delta, \ldots$ for any context. When Γ is $x_1 : A_1, \ldots, x_n : A_n$, by $PFV(\Gamma)$, we denote the free propositional variables in the set of formulas $\{A_1, \ldots, A_n\}$. A **sequent** is a triple, written $\Gamma \vdash t : A$, consisting of a context, a term, and a formula. A sequent $\Gamma \vdash t : A$ is **derivable** if it can be produced by the following inference rules.

$$\frac{}{\Gamma, x : A \vdash x : A}\ ax \qquad \frac{}{\Gamma \vdash c^A : A}\ ax$$

$$\frac{\Gamma, x : A \vdash t : B}{\Gamma \vdash \lambda x.t : A \to B} \to i \qquad \frac{\Gamma \vdash t : A \to B \quad \Gamma \vdash s : A}{\Gamma \vdash ts : B} \to e$$

$$\frac{\Gamma \vdash t : A}{\Gamma \vdash \Lambda X.t : \forall X.A}\ \forall i \text{ where } X \notin PFV(\Gamma) \qquad \frac{\Gamma \vdash t : \forall X.A}{\Gamma \vdash tY : A[X := Y]}\ \forall e$$

When we focus on term t in a sequent $\Gamma \vdash t : A$, we refer to it as "term t of A from (assumptions) Γ."

We consider the usual β-reduction rules for terms.

Definition 4. β**-reduction** $\to_\beta$ is defined as follows.

- $(\lambda x.t)s \to_\beta t[x := s]$ and $(\Lambda X.t)Y \to_\beta t[X := Y]$
  where each term of the left-hand side of $\to_\beta$ is called a **redex**.
- If $s \to_\beta t$ then, for any term u, $us \to_\beta ut$; $su \to_\beta tu$; and $\lambda x.s \to_\beta \lambda x.t$

A term t is in **normal form**, if t contains no redex.
The **multi-step** β**-reduction** $\twoheadrightarrow$ is the reflexive and transitive closure of $\to_\beta$.

3 Proof-theoretic validity of Prawitz

Validity of a term of an atom is defined by introducing an atomic base, which defines the derivability relation among atoms. Although production rules or higher-level inference rules are typically introduced to the atomic base, we consider our atomic base to consist simply of a set of term-constants for atoms, which are regarded as proper axioms. Thus, we do not assume any structure for our atomic base, which, according to Schroeder-Heister's terminology, is an atomic base of level 0.

Definition 5. An **atomic base** S is a set of term-constants for atoms. When $c^X \in S$, axiom $\Gamma \vdash c^X : X$ is called S-axiom.
A term t is **proof-term** when no term-constant appears in t other than $c^X \in S$.

We assume an atomic base S is fixed arbitrarily. From the natural deduction viewpoint, we regard a derivable term containing term-constants for $c^A \notin S$ as a proof in the process of being constructed. That is, axiom $\Gamma \vdash c^A : A$ requires a further proof of A, while a proof-term (without such term-constants) is considered a constructed proof.

The following definition of validity is essentially that of [16], although we do not consider any extension of our atomic base. See [14, 15, 5, 7] for proof-theoretic validity in the non-monotonic framework (without base extensions). The notion of proof-theoretic validity is firstly defined for "closed" terms. Then, validity of open terms containing free term-variables is defined by substituting valid closed terms for free term-variables appropriately.

Since $\mathsf{IL}^2\mathsf{at}$ includes term-constants, we extend the notion of validity, by including term-constants, resulting in q-validity (quasi validity). Then, validity is defined for proof-terms that do not contain term-constants, except for $c^X \in S$.

Definition 6. q-validity for term t is defined as follows.

1. A closed term t of atom X is q-valid *iff* $t \twoheadrightarrow s$ for normal s such that $\vdash s : X$ is derivable.
2. A closed term t of $B \to C$ is q-valid *iff* ts of C is q-valid for any q-valid closed term s of B.
3. A closed term t of $\forall X.B$ is q-valid *iff* tY of $B[X := Y]$ is q-valid for any Y.
4. An open term t of B from $x_1 : A_1, \ldots, x_n : A_n$ with $tFV(t) \subseteq \{x_1, \ldots, x_n\}$ is q-valid *iff* $t[x_1 := t_1, \ldots, x_n := t_n]$ of B is q-valid for any q-valid closed term t_i of A_i $(1 \leq i \leq n)$.

Definition 7. A term t is **valid** if t is a proof-term and is q-valid.

The main lemma to show the completeness is as follows, which is proved in [17].

Lemma 8. *The following holds.*

1. *For a closed $c^B T_1 \cdots T_n$ of A, where each T_i denotes a term or an atom and $n \geq 0$, if $c^B T_1 \cdots T_n \twoheadrightarrow s$ for normal s such that $\vdash s : A$ is derivable, then $c^B T_1 \cdots T_n$ of A is q-valid.*
2. *If a closed t of A is q-valid, then $t \twoheadrightarrow s$ for normal s such that $\vdash s : A$ is derivable.*

We obtain the following completeness theorem.

Theorem 9 (Completeness)**.** *If a term t of A from Γ is valid, then $t \twoheadrightarrow s$ for normal proof-term s such that $\Gamma \vdash s : A$ is derivable in* $\mathsf{IL}^2\mathsf{at}$.

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Conceptual completeness for geometric logic via ultraconvergence spaces

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The aim of this talk, based on [8], is to present a (strong) conceptual completeness theorem for geometric logic, generalizing Makkai-Lurie's analogous result for coherent logic [4, 3]. Towards this goal, we will introduce *ultraconvergence spaces*, a joint generalization of ultracategories and topological spaces inspired by Barr's description of the latter as *relational β-algebras*.

Background. A conceptual completeness theorem, for a logic, is a *reconstruction* result for syntax starting from semantics. More precisely, it amounts to recovering a theory in the logic from its class of models, and it typically requires to consider additional structure on the latter. For instance, in the context of classical propositional logic, where theories can be identified with their Lindenbaum-Tarski algebras, this takes the form of Stone's representation theorem for Boolean algebras: every Boolean algebra B is isomorphic to the algebra of continuous maps $\mathrm{Mod}(B) \to \mathbf{2}$, once the set $\mathrm{Mod}(B) \coloneqq \mathbf{BoolAlg}(B, \mathbf{2})$ is endowed with the *Stone topology*.

In [4], Makkai introduced *ultracategories* in order to prove a conceptual completeness theorem for first-order coherent logic. Intuitively, an ultracategory is a category endowed with abstract *ultraproducts*: the usual notion of ultraproducts then makes the category of models of a coherent theory into an ultracategory. This structure is what allows for the reconstruction theorem to go through: in Lurie's reformulation [3], identifying coherent theories with their *classifying toposes*, we have that a coherent topos $\mathcal{E}$ is equivalent to the category of functors $\mathrm{pt}(\mathcal{E}) \to \mathbf{Set}$ preserving ultraproducts in an appropriate sense.

The previous result crucially relies on *Łoś's theorem*: an ultraproduct of models of a coherent theory $\mathbb{T}$ is itself a model of $\mathbb{T}$. Categorically, this can be expressed by saying that, for each ultrafilter ν on a set I, the ultraproduct functor

$$\mathbf{Set}^I \xrightarrow{\prod_{i:\nu}(-)} \mathbf{Set},$$

witnessing **Set** as an ultracategory, is *coherent*, i.e. it preserves finite limits, regular epimorphisms, and finite unions of subobjects. Crucially, it does not preserve arbitrary unions of subobjects, meaning that Łoś's theorem fails for *geometric* logic. Thus, Makkai-Lurie's reconstruction result cannot be immediately extended to geometric theories.

Ultraconvergence spaces. The key intuition allowing us to extend Makkai-Lurie's theorem comes from [2]. In the (small and) discrete case, ultracategories recover compact Hausdorff spaces, which by Manes' theorem [5] coincide with the algebras for the ultrafilter monad $\beta\colon \mathbf{Set} \to \mathbf{Set}$. This fact, originally noticed by Lurie, is especially evident in Rosolini's definition of ultracategories [6] as pseudoalgebras for the *ultracompletion* pseudomonad $\beth\colon \mathbf{CAT} \to \mathbf{CAT}$: for a category C, objects of $\beth C$ are formal ultraproducts (or *ultrafamilies*) in C, i.e. tuples $(c_i)_{i\in I}$ of objects of C together with an ultrafilter ν on the index set I, which an algebra functor $\beth C \to C$ maps to 'actual' ultraproducts in C.

*Presenting author

In [2], Barr extended Manes' description of compact Hausdorff spaces as β-algebras to all topological spaces by allowing for *relational β-algebras*: that is, lax algebras for a (lax) monad extending β to the 2-category **Rel** of sets and relations. In other words, an arbitrary topology on X is described by a suitable relation $\beta X \nrightarrow X$ – i.e. a function $X \times \beta X \to \mathbf{2}$ – which we can think of as describing a notion of convergence between ultrafilters on X and points of X. In these terms, a topology is compact and Hausdorff if and only if the convergence relation is actually a function $\beta X \to X$, intuitively computing 'limits' of ultrafilters.

The role of relations, between categories, is played by *profunctors*: following Barr's blueprint, we introduce ultraconvergence spaces [8, 7] as a profunctorial generalization of ultracategories. Intuitively, these categorify topological spaces by replacing a convergence relation $X \times \beta X \to \mathbf{2}$ with a profunctor $C^{\mathrm{op}} \times \text{ß}C \to \mathbf{Set}$ which associates, to an object c and an ultrafamily $(d_i)^{\nu}_{i\in I}$ in C, a set of *ultra-arrows* $c \prec (d_i)^{\nu}_{i\in I}$ acting as 'witnesses' of the convergence of $(d_i)^{\nu}_{i\in I}$ to c. In the same spirit we introduce *continuous maps* of ultraconvergence spaces, categorifying the topological notion, and suitable 2-cells between them, obtaining a 2-category **UltSp**.

Conceptual completeness. Based on [8], we show how ultraconvergence spaces allow for a conceptual completeness theorem for geometric logic. Indeed, the category $\mathrm{Mod}(\mathbb{T})$ of models of a geometric theory $\mathbb{T}$ in a signature Σ carries a canonical structure of ultraconvergence space, where ultra-arrows $M \prec (N_i)^{\nu}_{i\in I}$ are Σ-structure homomorphisms from the model M into the ultraproduct $\prod_{i:\nu} N_i$, even when the latter fails to be a model of $\mathbb{T}$ itself.

Identifying $\mathbb{T}$ with its classifying topos $\mathcal{E}$, the ultraproduct $\prod_{i:\nu} N_i$ of an ultrafamily $(N_i)^{\nu}_{i\in I}$ of points of $\mathcal{E}$ can be identified with the functor

$$\mathcal{E} \xrightarrow{\langle N_i\rangle_{i\in I}} \mathbf{Set}^I \xrightarrow{\prod_{i:\nu}(-)} \mathbf{Set}$$

which need not be itself a point of $\mathcal{E}$: in these terms, Σ-structure homomorphisms $M \to \prod_{i:\nu} N_i$ correspond simply to natural transformations $M \Rightarrow \prod_{i:\nu} N_i$, making $\mathrm{pt}(\mathcal{E})$ into an ultraconvergence space. Our theorem reads then as follows.

Theorem. *Let $\mathcal{E}$ be a topos with enough points. Then, $\mathcal{E} \simeq \mathbf{UltSp}(\mathrm{pt}(\mathcal{E}), \mathbf{Set})$.*

Time permitting, we will comment on [1], joint work of the presenting author with Quentin Aristote, where ultraconvergence spaces are recovered algebraically by extending the pseudomonad ß to profunctors.

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Applications of Topology via Lawvere Doctrines

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Abstract

Localic and realizability toposes are two central classes of toposes in categorical logic. One of the main differences between them is that realizability toposes provide fundamental examples of elementary toposes that are not Grothendieck toposes.

To investigate the common features of these two families, Hyland, Johnstone, and Pitts introduced the tripos-to-topos construction in [1]. This construction associates a topos with a suitable kind of Lawvere doctrine, called a *tripos*, and shows that both localic and realizability toposes arise as instances of this construction. This represented the first major step toward a unified treatment of these two important classes of toposes.

Despite this achievement, the tripos-to-topos construction seems to obscure the original geometric meaning of the notion of *sheaf*. Indeed, sheaves are traditionally understood as a topological concept, whereas the tripos-to-topos construction is purely logical, see e.g. [6], making it unclear how the topological intuition behind sheaves can be recovered.

The main goal of this talk is to bridge this gap by showing that the notion of sheaf naturally emerges from the tripos-to-topos construction through a suitable generalization of Lawvere-Tierney topologies to the setting of Lawvere doctrines and triposes.

More precisely, we show that every tripos whose base category has weak dependent products and a generic proof admits a rich family of topologies. These generalize several familiar examples from topos theory, including sheafification, double-negation, and canonical topologies.

The key idea is that each such topology is induced by a suitable quantifier completion of the original tripos.

For example, ordinary localic sheafification corresponds to the completion which freely adds existential quantification along every morphism of the base

category [10]. The original tripos can then be recovered as the tripos of j-sheaves for the topology induced on its existential completion [8].

Furthermore, every topology on a tripos induces a Lawvere-Tierney topology on the associated topos via the tripos-to-topos construction.

In particular, given a topology on a tripos, the topos generated by the original tripos is equivalent to the topos of j-sheaves for the corresponding Lawvere-Tierney topology.

The localic sheafification, double-negation, and canonical topologies can be then presented as particular cases of our results.

Relevant examples include all toposes arising from Set-based triposes, such as localic toposes, the Effective topos, the Modified Realizability topos [2], the Extensional Realizability topos [3], the Dialectica topos [4], and the Krivine topos [5]. As a further example beyond the Set-based setting, we consider the topos of extended Weihrauch degrees recently introduced in [9].

This talk is based on joint work with Maria Emilia Maietti [8] and Samuele Maschio [9].

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Asymptotic Ramsey Theory

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1 Abstract

Arithmetic Ramsey theory is the branch of combinatorics devoted to the study of patterns which cannot be avoided taking a finite partition of the natural numbers. To give a flavor, finite partitions are usually called *finite colorings* and a pattern contained within a single color, well, *monochromatic*; configurations that always appear monochromatically are called *partition regular (PR)*.

The history of Ramsey theory goes back to the first decades of the last century, with the seminal works of Schur (the pattern $\{x, y, x+y\}$ is PR, [9]), Rado (full characterization of PR linear equations, [7]), Van der Waerden (arbitrarily long arithemtic progressions are PR, [10]) and, of course, Ramsey ([8]), whose theorem goes far beyond the line of Arithmetic Ramsey theory and is considered a staple for logicians.

This subject has attracted many mathematicians during the years, due to the combination of methods that can be adopted in order to prove that a particular pattern can or cannot be avoided: pure combinatorics, topological dynamics, ergodic theory, functional analysis; and in the realm of logic, ultrafilters, nonstandard analysis and model theory are strong tools that in the last twenty years have been widely adopted. In this talk, we will focus on the nonstandard approach.

We consider a sufficiently saturated elementary extension ${}^*\mathbb{N}$ of the natural numbers, where $\mathbb{N}$ is equipped with its full structure, i.e., we have a symbol for every subset of every power $\mathbb{N}^n$. Given a set A, we call ${}^*A$ the interpretation of (the symbol for) A in ${}^*\mathbb{N}$. We say that two numbers α and β are equivalent if they have the same type over $\mathbb{N}$; equivalently, if they generate the same ultrafilter over $\mathbb{N}$; more concretely, if for every subset A of $\mathbb{N}$

$$\alpha \in {}^*A \Leftrightarrow \beta \in {}^*A.$$

This equivalence plays a fundamental role in determining whether a configuration is partition regular or not, due to the following well-known theorem.

Fact 1.1. *A finite pattern is partition regular if and only if there exist equivalent numbers in ${}^*\mathbb{N}$ disposed on that pattern.*

This implies that we can disprove partition regularity showing that it is not possible to find equivalent numbers satisfying it.

The main patterns we do care here are solutions to polynomials. We say that a polynomial is partition partition regular if in any finite coloring of $\mathbb{N}$ we can find a monochromatic solution, so by the theorem mentioned above we know that a polynomial P is PR if and only if there exist $\alpha_1 \sim \cdots \sim \alpha_k$ such that $P(\alpha_1, \ldots, \alpha_k)=0$.

We have talked about Ramsey theory; if you look at the title once again, you should expect now to see what *Asymptotic* stands for. Ok then: in ${}^*\mathbb{N}$ there is a well-defined notion of Archimedean classes, where we say that two numbers α and β are in the same Archimedean class (written $\alpha \asymp \beta$) if for some natural number n,

$$\frac{\alpha}{n} \leq \beta \leq n\alpha.$$

Just to give an idea: all natural numbers are in the same class, α and 4α are in the same class, α and α^2 are never in the same class, if α is infinite. This notion might seem very weak; the magic happens when it interacts with equivalence.

Fact 1.2 ([3], Lemma 5.5)**.** *Let α, β be two elements in ${}^*\mathbb{N}$ such that $\alpha \sim \beta$ and $\alpha \asymp \beta$. Then $st(\frac{\alpha}{\beta}) = 1$, where $st : {}^*\mathbb{R} \to \mathbb{R}$ is the standard part.*

So, if two equivalent numbers are close, they must in fact be extremely close.

Building on this fact, in [6] we develop a theory that cares about relative largeness: we say that an equation $E(x_1, \ldots, x_n) = 0$ is asymptotic PR in $I_1, \ldots, I_k$ (this is a partition of the variables) if for every finite coloring of $\mathbb{N}$ and for every $N \in \mathbb{N}$ there are monochromatic solutions $a_1, \ldots, a_n$ such that:

1. for all $r \leq s$, for all $i, j \in I_r$ $|\frac{a_i}{a_j} - 1| < \frac{1}{N}$;
2. for all $r, t \in \{1, \ldots, s\}$, for all $i \in I_r, j \in I_t$, if $r < t$ then $Na_j < a_i$.

We translate this in nonstandard terms and relate it to the notion of Archimedean class; this shows that any partition regular equation can be seen as an asymptotic partition regular equation, for some partitions of its variables.

We then apply this theory to the case of polynomials, and what is impressive is that all (known) negative results about polynomials can be found using asymptotic considerations. For example, necessity condition in Rado's theorem follows immediately from a simple argument, which can be generalized to many different situations. The easy argument informally reads like this: if the polynomial must be zero, then the monomials in the largest Archimedean class better sum to something way smaller, for the polynomial to have a chance of being zero.

Using this basic idea, we prove that a wide class of polynomials cannot be PR; in addition to Rado, we also re-obtain the necessary conditions from [1] and [2], and generalize [5] via the following result.

Theorem 1.3. *Let $n \geq 2$, let $a, b \in \mathbb{Z}$ and let $P(z) \in \mathbb{Z}[z]$ be a polynomial of degree k. If $k \notin \{n, n-1\}$ then $ax^n + by^n = P(z)$ is not PR, if not trivially (i.e. with constant solutions).*

In particular, we see that polynomials of the form $x^n + y^m - z^k$ can be partition regular only if (up to symmetries) $n = m$ and $k = n, n-1$, the case $k = n$ being trivially excluded for $n > 2$ by Fermat's last theorem. We leave open the case $k = n - 1$.

More recently, asymptotic methods have also been applied in [4] to deduce the non-PR of certain configurations, and they appear to provide a simple and effective way to obtain negative results in arithmetic Ramsey theory. We hope to illustrate this during the talk.

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The Internal Modal Logic of Forcing

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Abstract

Forcing is usually studied *externally*: one starts with a ground model of set theory, passes to generic extensions or to ultrafilter quotients of a Boolean-valued universe, and asks which sentences become true in those classical models. The guiding claim of this paper is that a substantial modal structure is already present *inside* a single Boolean-valued model, before any passage to a family of external worlds. If B is a complete Boolean algebra and $V^{(B)}$ is the corresponding Boolean-valued universe, then the elements of B may be read not merely as truth values but as local standpoints from which truth is assessed. This yields a natural internal modal semantics whose accessibility relation is Boolean compatibility:

$$aRb \quad \Longleftrightarrow \quad a \wedge b \neq 0.$$

The relation is always symmetric, but in general it is not transitive and it fails to be globally reflexive because 0 is isolated. The resulting logic is therefore sharply different from the familiar modal logic of forcing extensions.

The basic semantic idea is simple. Every set-theoretic sentence σ has a Boolean truth value $[\![\sigma]\!]_B \in B$. We interpret a Boolean value $b \in B$ as a world, and we say that σ holds at b just in case $b \leq [\![\sigma]\!]_B$. Modal operators are then evaluated over the compatibility frame (B, R) in the usual Kripke fashion. In this way, internal necessity and possibility become modalities of *co-consistency* inside the Boolean algebra of truth values itself. The project is not merely terminological: it produces a precise bridge between Boolean-valued truth, ultrafilter quotients, and forcing-style possibility, and it isolates a modal theory that is intrinsic to Boolean-valued semantics.

1 Internal Semantics and Its Conceptual Content

The first contribution is the identification of a robust internal Kripke semantics native to $V^{(B)}$. A state $b \in B$ is best understood as a local perspective from which a sentence counts as true when its Boolean value extends b. The accessibility relation R records compatibility rather than extension, so the modal geometry is horizontal rather than upward-directed. This is the crucial difference from external forcing semantics. Instead of moving from one model of set theory to a stronger forcing extension, one remains inside a single Boolean-valued universe and asks whether a sentence can be realized at some compatible Boolean state.

This interpretation captures the intuitive thought that Boolean truth values encode more than degrees of truth: they organize a space of mutually compatible realizations. The internal point of view therefore reveals a modal dimension already latent in standard Boolean-valued semantics. The main results of the paper make this claim mathematically exact.

2 Ultrafilter Quotients as Witnesses to Internal Possibility

The central theorem identifies internal possibility with realizability in a classical quotient.

Theorem 1 (Ultrafilter characterization of possibility). *Let B be a complete Boolean algebra, let p be a non-modal set-theoretic sentence, and let $b \in B$. Then*

$$M_B, b \models \Diamond p \quad \Longleftrightarrow \quad \textit{there is an ultrafilter } U \textit{ on } B \textit{ with } b \in U \textit{ and } V^{(B)}/U \models p.$$

This theorem gives the conceptual core of the paper. It shows that internal possibility is not an ad hoc modal decoration placed on top of Boolean-valued semantics; rather, it exactly tracks the existence of a classical realization compatible with the current Boolean condition. The proof is guided by the fact that $\Diamond p$ at b means there is some c with $b \wedge c \neq 0$ and $c \leq [\![p]\!]_B$. Any such $b \wedge c$ can be extended to an ultrafilter U, and the quotient theorem then transfers $[\![p]\!]_B \in U$ to truth of p in $V^{(B)}/U$. Conversely, if a quotient by an ultrafilter containing b satisfies p, then $[\![p]\!]_B \in U$, so b is compatible with a state forcing p, which is exactly the internal modal clause for $\Diamond p$.

At the top element 1_B, this recovers the familiar forcing-theoretic slogan that a sentence is possible exactly when it is true in some suitable classical collapse. The theorem also explains where the internal and external perspectives agree: for non-modal set-theoretic sentences, possibility in the internal frame coincides with forceability below a Boolean condition. The divergence only appears once nested modalities enter the picture.

3 General Validities and the Role of the Zero State

Once the compatibility semantics is fixed, one can ask which modal principles hold uniformly across complete Boolean algebras. Because compatibility is symmetric, the system validates the axiom B:

$$\varphi \to \Box\Diamond\varphi.$$

It also validates the normal axiom K, the usual duality of $\Box$ and $\Diamond$, and additional principles reflecting the structure of compatibility semantics, such as

$$\Diamond\Box\varphi \to \Box\Diamond\varphi \qquad \text{and} \qquad \Diamond\Box\varphi \to \varphi.$$

What fails is equally important. The full state space B contains the isolated point 0, and this single degenerate state has decisive modal consequences.

Theorem 2 (Failure of reflexive principles on the full frame). *In the full-state semantics on B, axioms implying reflexivity—in particular T $(\Box\varphi \to \varphi)$—need not be valid. At the state 0, every formula of the form $\Box\varphi$ holds vacuously, while every formula of the form $\Diamond\varphi$ fails.*

The presence of 0 shows that the internal modal logic is not simply another presentation of ordinary forcing potentialism. Externally, trivial forcing guarantees reflexivity. Internally, the isolated zero state destroys reflexivity and prevents a uniform algebra-independent logic on all states. Distinct Boolean algebras can therefore support different full-state modal theories. The lesson is that 0 is not a harmless boundary point but a mathematically active source of non-normal behavior.

4 Parameters, Realizability of Boolean Values, and Exact Completeness on B^+

A second major theme of the paper is the distinction between parameter-free and parameterized semantics. In atomic complete Boolean algebras with more than two elements, parameter-free set-theoretic sentences do not realize arbitrary Boolean values: they collapse to truth values 0 and 1 only. Without parameters, the language simply cannot see enough of the internal algebraic structure. The paper shows that this obstacle disappears as soon as one expands the language by constants from $V^{(B)}$.

Theorem 3 (Realization of arbitrary Boolean values with parameters). *For every complete Boolean algebra B and every $a \in B$, there is a sentence in the expanded language with parameters from $V^{(B)}$ whose Boolean truth value is exactly a.*

This realization theorem allows one to pass from set-theoretic sentences to a translation semantics in which propositional variables range over arbitrary Boolean-valued sentences. On the nonzero part

$$B^+ = B \setminus \{0\},$$

the compatibility relation becomes reflexive as well as symmetric, and the modal behavior stabilizes completely.

Theorem 4 (Exact completeness on the nonzero frame). *The formulas valid in all translation models based on B^+ are exactly the theorems of KTB. Equivalently, once arbitrary Boolean values are available and the zero state is removed, internal compatibility semantics is precisely the modal semantics of reflexive symmetric frames.*

This is the paper's main completeness result. It isolates the exact internal modal logic of compatibility on nonzero states and shows that no stronger normal system is forced by Boolean-valued semantics in general. The proof combines the realization theorem above with a frame-theoretic analysis: every counterexample to a non-KTB formula on a reflexive symmetric frame can be simulated inside an appropriate nonzero Boolean compatibility frame. By contrast, if one reinstates 0 in the translation semantics, the valid formulas are no longer closed under uniform substitution, so there is no normal modal logic capturing the full translated semantics.

5 Fixed Boolean Algebras and Comparison with the External Forcing Modality

The final structural result concerns the dependence of the logic on the underlying algebra.

Theorem 5 (Finite/infinite dichotomy for fixed algebras). *If B is an infinite complete Boolean algebra, then the modal logic of translation models on B^+ is exactly KTB. If B is finite, the logic can be strictly stronger. In particular, the two-element algebra collapses modality to ordinary truth at its unique nonzero state, while the four-element algebra already witnesses the failure of stronger principles such as 4 and 5 once parameters are allowed.*

The infinite case reflects the fact that every infinite complete Boolean algebra contains enough internal structure to simulate the finite reflexive symmetric frames needed for KTB-completeness. Finite algebras, by contrast, lack the combinatorial room required to realize all such frames, so extra validities may appear.

The paper closes by contrasting the internal semantics with the external modal logic of forcing in the sense of Hamkins and Löwe. Externally, the worlds are models of set theory ordered by forcing extension; the relation is reflexive, transitive, and directed, yielding the modal logic S4.2. Internally, the worlds are Boolean truth values inside a single $V^{(B)}$, and accessibility is compatibility. Even on B^+, this relation is generally non-transitive, so the logic stops at KTB. The two viewpoints therefore coincide on the bare possibility of non-modal set-theoretic sentences, by the ultrafilter theorem above, but they diverge as soon as one iterates modal operators. This divergence is not a defect. Rather, it reveals that Boolean-valued models support two genuinely different modal perspectives: an *external* modality generated by passage

to forcing extensions and an *internal* modality generated by the compatibility geometry of truth values.

Conclusion

The paper identifies and analyzes an overlooked modal aspect of Boolean-valued semantics. Its main results show that: (i) every complete Boolean algebra carries a natural internal Kripke frame; (ii) internal possibility for non-modal set-theoretic sentences is exactly realizability in an ultrafilter quotient; (iii) after allowing parameters and restricting to nonzero states, the exact modal logic is KTB; and (iv) for fixed algebras one obtains a sharp finite/infinite dichotomy, together with a clear contrast to the external forcing logic S4.2. The upshot is that forcing admits not only the familiar modal analysis by moving among models, but also an intrinsic modal analysis inside a single Boolean-valued universe.

Adapted Pairs and Logical Relations in Linear Logic

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Abstract

It is folklore that there is a tight link between biorthogonal-closure constructions, familiar to practitioners of linear logic (LL), and logical relations, as they appear in, for instance reducibility-candidate flavoured proofs of normalisation for typed lambda calculi. One manifestation of this is in the technique of adapted pairs, due to Krivine, which weakens the extensionality condition defining a logical relation to facilitate the required induction over types. In this technique, the full logical-relation condition appears as a kind of fixed-point of the adapted-pair condition. We aim to reformulate the duality underlying the definition of adapted pairs using linear logic, and connect this duality to techniques for proving denotational completeness (or definability) in models of LL.

1 Background

A logical relation (sometimes logical predicate, in the unary case we consider) constructs sets of terms based on their extensional behaviour. For example, if $\mathcal{S}, \mathcal{R} \subset \Lambda$ are sets of untyped λ-terms, then their *Kleene implication* is defined as

$$\mathcal{S} \to \mathcal{R} \doteq \{t \mid \forall s \in \mathcal{S}, ts \in \mathcal{R}\}.$$

This definition appears, broadly speaking, in two contexts. In *realisability*, it gives precise content to the Brouwer-Heyting-Kolmogorov interpretation of intuitionistic logic: following Kleene [Kle45], a realiser of the formula $A \to B$, is a term that takes, via function application, realisers of A to realisers of B. On the other side, arguments using *reducibility candidates*, or *computability predicates* [Tai67, Gal89], first prove the required property for all terms matching an extensional specification — defined using a logical relation — then prove that the class of terms in question match an appropriate specification.

In the context of "reducibility-style" proofs, the extensional specification is provided by types, and the induction over types is sometimes facilitated by the technique of *adapted pairs*, due to Krivine [Kri93]. As a concrete example, consider the hierarchy $\mathbb{T}$ of simple types over base types $\mathbb{B}$. Then, having fixed $R_o \subset \Lambda$ for $o \in \mathbb{B}$, we can define the "reducible terms" or "realisers" of every type $\sigma \in \mathbb{T}$ by induction:

$$R_{\sigma\to\tau} \doteq R_\sigma \to R_\tau. \tag{1}$$

Let us call a pair (M_σ, N_σ) of $\mathbb{T}$-indexed families of subsets of Λ *adapted* if

$$M_{\sigma\to\tau} \subset N_\sigma \to M_\tau \subset M_\sigma \to N_\tau \subset N_{\sigma\to\tau}. \tag{2}$$

This definition is designed so that if $M_o \subset R_o \subset N_o$ at base types o, then $M_\sigma \subset R_\sigma \subset N_\sigma$ for every type. Instantiating M_σ as a collection *neutral* terms $xt_1 \cdots t_n$ and N_σ as the collection of strongly normalising terms, (M, N) is adapted and one can use this technique to provide (yet another) proof of strong normalisation for the simply-typed lambda calculus.

It is evident from this presentation that eq. (2) is a weakening of the definition eq. (1): that a family R_σ defines a logical predicate if and only if (R_σ, R_σ) is an adapted pair. With the biorthogonal constructions of linear logic (LL) in mind, we might expect to find a duality for which logical predicates R_σ are fixed points. Indeed this perspective is implicit in presentations of reducibility candidates for linear logic [Gir87, Acc13], and has been made explicit for the lambda calculus in [MM11]. In a realisability setting, a similar perspective is put forward in [RSdF24].

2 Kripke Logical Predicates

Logical relations also arise as a technique for demonstrating definability results: in a slightly generalised form they were used in [JT93] to characterise those morphisms in a model of the lambda calculus which arise as denotations of terms, and this technique was extended to linear logic in [Str99].

To this end we must consider open terms, so that the logical predicates will be indexed by a type σ and a context Γ, in order to characterise morphisms definable by terms t typed as $\Gamma \vdash t : \sigma$. This leads to the definition of *Kripke logical relations*. Let Tm be the category of $\mathbb{T}$-terms — the free cartesian-closed category (CCC) over the set $\mathbb{B}$ — and Ctx the wide subcategory of structural rules. Now fixing a CCC C and a function $\mathbb{B} \to \mathsf{C}$, we have an interpretation $i : \mathsf{Tm} \to \mathsf{C}$. Kripke logical relations live in the Artin-Wraith gluing [AGV06, Wra74] G of C to the category $\hat{\mathsf{Ctx}}$ of presheaves on Ctx, along the extended Yoneda functor $\text{ɏ}^i : \mathsf{C} \to \hat{\mathsf{Ctx}}$, defined by

$$\text{ɏ}^i A \doteq \mathsf{C}(i-, A).$$

More explicitly:

Definition 2.1. A *Kripke predicate* on $A \in \mathsf{C}$ is a map R from contexts Γ to subsets of $\mathsf{C}(i\Gamma, A)$ so that, if $w : \Gamma' \to \Gamma$ is a morphism in Ctx, then

$$w^*(R\Gamma) \doteq \{f \circ iw \mid f \in R\Gamma\} \subset R\Gamma'.$$

Kripke predicates (for varying A) form a cartesian-closed category G, whose morphisms $(A, R) \to (B, S)$ are those C-maps $R \to S$ which preserve the predicate. The product is pointwise, and the internal hom recovers the Kleene arrow: given relations R on A and S on B and a context Γ, the predicate $R \to S$ on $A \to B$ is

$$(R \to S)\Gamma = \{f \in \mathsf{C}(i\Gamma, A \to B) \mid \forall w : \Gamma' \to \Gamma, a \in R\Gamma', \mathrm{ev}(f \circ iw, a) \in S\Gamma'\}.$$

This mimics our earlier definition, taking account of the fact that the "test" argument a may be defined only in a larger context Γ' — it is worth noting however that this fact is a consequence of the cartesian-closed structure on G, originally defined in topos theory.

The prototypical example of a Kripke predicate is the family of definable morphisms D_σ into a type σ, viz:

$$D\Gamma \doteq \{f \in \mathsf{C}(i\Gamma, i\sigma) \mid \exists t \in \mathsf{Tm}(\Gamma, \sigma),\ it = f\}.$$

Indeed, Kripke logical predicates characterise definable morphisms: the result of [JT93] amounts to showing that D_σ is logical, in the sense that the syntactic and semantic constructions of the arrow type agree:

$$D_{\sigma \to \tau} = D_\sigma \to D_\tau.$$

In the guise of *full completeness* [AJ94], this is also presented in [Tay99, Ali95].

As observed (at least) by Fiore [Fio22], the technique of adapted pairs scales up to the presheaf setting. Indeed, for A_σ, B_σ indexed families of Kripke predicates on $i\sigma$, we can upgrade the definition of eq. (2) to inclusions of presheaves

$$A_{\sigma\to\tau} \longhookrightarrow (B_\sigma \to A_\tau) \longhookrightarrow (A_\sigma \to B_\tau) \longhookrightarrow B_{\sigma\to\tau},$$

and precisely the same argument applies: if R_o is a Kripke predicate on each base type o, and $A_o \hookrightarrow R_o \hookrightarrow B_o$, then the (unique) extension of R to $\mathbb{T}$ satisfies $A_\sigma \hookrightarrow R_\sigma \hookrightarrow B_\sigma$ for all σ. This becomes particularly interesting if we generalise the inclusions to morphisms: for instance replacing A and B with presheaves M, N of neutral and *normal* (rather than normalising) terms, constructing a morphism $R_\sigma \to N_\sigma$ in an appropriate category provides an algorithm implementing normalisation [AHS95, Fio22].

3 Linear Logic

A similar technique is applied by Streicher to characterise which morphisms in a model C of (classical) linear logic (LL) arise as denotations of proofs [Str99]. Here the situation is not quite as simple: the glued category G may not be $*$-autonomous, that is, the interpretation in presheaves may not respect the duality of linear logic.

Suppose that we interpret the type $\bot$ as a Kripke predicate

$$\mathcal{B} \longhookrightarrow \mathsf{C}(i-, i\bot).$$

Then if we have interpreted σ and $\sigma^\bot$ as presheaves R and $R^\bot$, for any $a \in R\Gamma$ and $b \in R^\bot\Gamma$ we must have $\mathrm{cut}(a,b) \in \mathcal{B}(\Gamma \otimes \Delta)$. If we take this requirement as a definition, and ensure at every step that the interpretation R_σ of any type σ satisfies $R_\sigma^{\bot\bot} = R_\sigma$, we obtain the following.

Definition 3.1 ([Str99] Definition 2.5)**.** Fixing R_o for $o \in \mathbb{B}$, and an interpretation $\mathcal{B} = R_\bot$, R_σ is defined inductively by

$$R_{\sigma^\bot}\Delta = \{b \in \mathcal{C}(i\Delta, A^\bot) \mid \forall \Gamma, a \in R\Gamma,\ \mathrm{cut}(a,b) \in \mathcal{B}(\Gamma \otimes \Delta),\}, \tag{3}$$

$$R_{\sigma\otimes\tau}\Gamma = \left\{(a \otimes b) \circ iw \mid a \in R_\sigma\Delta, b \in R_\tau\Delta', w \in \mathsf{Ctx}(\Gamma, \Delta \otimes \Delta')\right\}^{\bot\bot}. \tag{4}$$

As for conventional logical predicates, Streicher's definition characterises the "proof objects" in C. The denotation of any proof satisfies all Kripke predicates arising from this inductive construction, and if we set

$$\mathcal{B}\Gamma = D_\bot\Gamma = \{f \in \mathsf{C}(i\Gamma, i\bot) \mid \exists t \in \mathsf{Tm}(\Gamma, \bot),\ it = f\},$$

and $R_o = D_o$, R_σ contains exactly the denotations of proofs. However, the correspondence between syntax and semantics is not as close: G is symmetric-monoidal-closed, but the correct tensor product is that of eq. (4) *without* the double-orthogonal (the Day convolution product [Day07]) — that is, the interpretation of types (formulas) as Kripke predicates no longer provides a model of LL[1]. Interestingly, the convolution product recovers the "splitting of the context" necessary to proving a tensor in linear logic [Has99].

To obtain a bona-fide model of linear logic, one solution due to Hyland and Tan [Tan97, HS03] is to define for each $A \in \mathsf{C}$ two predicates, one "covariant" and one "contravariant", so that the duality swaps the two. Many models of LL arise as variations of this

[1] Hasegawa observes that it can in fact be seen as such, by moving to the category of algebras for a double-dualisation monad [Has99, Example 3.9]

construction, including coherence and finiteness spaces [Ehr05, Gir87], and the construction as a whole can also be interpreted in the style of Gödel's Dialectica interpretation [AF95] and classical realisability [Kri04], where the covariant part represents proofs of A, and the contravariant part counter-proofs [Tan97, §1.2]

4 Adapted Pairs in the Glued Category

We aim, in ongoing work, to connect the ways that categories of logical relations are made self-dual, in order to be models of linear logic, with Krivine's technique of adapted pairs. The latter is well-established and understood, and, as sketched at the end of Section 2, has been generalised to provide normalisation-by-evaluation results.

At present it suffices to show how Hyland and Tan's construction sandwiches self-dual Kripke predicates between a kind of adapted pair. Let C be $*$-autonomous, and fix a subpresheaf $\mathcal{O} \hookrightarrow \mathsf{C}(i-, \bot)$ in $\hat{\mathsf{Ctx}}$. Following [HS03, §5] this defines an *orthogonality* on maps $f : i\Gamma \to A$ and $g : \Delta \otimes A \to \bot$ by

$$f \perp_{\mathcal{O}} g \iff g \circ (1 \otimes f) \in \mathcal{O}(\Delta \otimes \Gamma).$$

For the tensor, evidently

$$R_\sigma \otimes R_\tau \longhookrightarrow R_{\sigma\otimes\tau} = (R_\sigma \otimes R_\tau)^{\perp_{\mathcal{O}}\perp_{\mathcal{O}}}.$$

On the other hand, $\sigma \otimes \tau$ is dual to both $\sigma \multimap \tau^\perp$ and $\tau \multimap \sigma^\perp$. If $a \in R_\sigma\Delta$ and $b \in R_\tau\Delta'$, then for any $f \in (R_\sigma \multimap R_\tau^{\perp_{\mathcal{O}}})\Gamma$, the cut of f against $a \otimes b$ is

$$\Gamma \otimes \Delta \otimes \Delta' \xrightarrow{f\otimes a\otimes 1} (R_\sigma \multimap R_\tau^{\perp_{\mathcal{O}}}) \otimes R_\sigma \otimes \Delta' \xrightarrow{\mathrm{ev}\otimes b} R_\tau^{\perp_{\mathcal{O}}} \otimes R_\tau.$$

(Of course, the argument swapping the roles of σ and τ works just as well.) As such $R_\sigma \multimap R_\tau^{\perp_{\mathcal{O}}} \hookrightarrow (R_\sigma \otimes R_\tau)^{\perp_{\mathcal{O}}}$, so finally

$$R_\sigma \otimes R_\tau \longhookrightarrow R_{\sigma\otimes\tau} \longhookrightarrow (R_\sigma \multimap R_\tau^{\perp_{\mathcal{O}}})^{\perp_{\mathcal{O}}}.$$

This is essentially the slack orthogonality subcategory of Hyland and Schalk's double gluing construction, generalised to the presheaf setting. This argument suggests that the lower part of a classical adapted pair comes from the (non-self-dual) interpretation of LL in the glued category G, and the upper part is the dual (with respect to the orthogonality in G), of the interpretation of the dual formula. As suggested above, the self-dual *definable elements* live in between, as fixed points of the duality on G.

The construction of a generalised adapted pair we have sketched here connects to the classical version if we set $\mathsf{C} = \mathsf{Tm}$, the syntactic model, and $\mathcal{O}$ to be the presheaf of strongly normalising terms, that is

$$\mathcal{O}\Gamma = \{\vdash t : \Gamma^\perp \mid t \text{ is strongly normalising}\}.$$

Then we recover the upper component N of the pair in eq. (2) (the presheaf of strongly normalising terms) in the LL setting as (cf: [Str99, Lemma 4.1])

$$\begin{aligned}\{\Gamma \vdash \pi : A \mid \pi \text{ strongly normalising}\} &= \{\Gamma \vdash \pi : A \mid \mathrm{cut}(\pi, \mathrm{ax}_{A^\perp}) \text{ strongly normalising}\} \\ &= \{\mathrm{ax}_{A^\perp}\}^{\perp_{\mathcal{O}}}.\end{aligned}$$

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